COMPACT GENERATION OF THE CATEGORY OF D-MODULES ON THE STACK OF G-BUNDLES ON A CURVE

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Contents

Date: June 4, 2012.

Introduction

0.1. **The main result.** Throughout the paper we will be working over an algebraically closed field k of characteristic 0.

0.1.1. The principal goal of this paper is to prove the following theorem:

Theorem 0.1.2. *Let X be a smooth complete connected curve over k and let $\mathcal{Y} = \mathrm{Bun}_G$ denote the moduli stack of principal G-bundles on X, where G is a connected reductive group. Then the DG category* $\text{D-mod}(\mathrm{Bun}_G)$ *of D-modules on* Bun_G *is compactly generated.*

For the reader's convenience we shall review the theory of DG categories, and the notion of compact generation in Sect. 0.8.

Essentially, the property of compact generation is what makes a DG category manageable.

0.1.3. The above theorem is somewhat surprising for the following reason.

It is known that if an algebraic stack $\mathcal{Y}$ is quasi-compact and the automorphism group of every field-valued point of $\mathcal{Y}$ is affine, then the DG category $\text{D-mod}(\mathcal{Y})$ is compactly generated. This result is established in [DrGa, Theorem 0.2.2]. In fact, the compact generation of $\text{D-mod}(\mathcal{Y})$ for most stacks $\mathcal{Y}$ that one encounters in practice is much easier than the above-mentioned theorem of [DrGa]: it is nearly obvious for stacks of the form Z/G, where Z is a quasi-compact scheme and H an algebraic group acting on it.

However, if $\mathcal{Y}$ is not quasi-compact then $\text{D-mod}(\mathcal{Y})$ does not have to be compactly generated. We will exhibit two such examples in Sect. 7.1; in both of them $\mathcal{Y}$ will actually be a smooth non quasi-compact scheme (non-separated in the first example, and separated in the second one).

0.1.4. So, the compact generation of $\text{D-mod}(\mathcal{Y})$ encodes a certain geometric property of the stack $\mathcal{Y}$. We do not know how to formulate a necessary and sufficient condition for $\text{D-mod}(\mathcal{Y})$ to be compactly generated.

But we do formulate a sufficient one. We call it "truncatibility"; see Definition 4.1.6. The idea is that $\mathcal{Y}$ is truncatable if it can be represented as a union of quasi-compact open substacks U_α such that the direct image functor

$$(j_\alpha)_* : \text{D-mod}(\mathcal{U}_\alpha) \to \text{D-mod}(\mathcal{Y}),$$

has a particularly nice property of explained below.

0.2. **Truncativeness, co-truncativeness and truncatabilty.**

0.2.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack with affine automorphism groups of points, and let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be a closed embedding. By [DrGa, Theorem 0.2.2], both categories $\text{D-mod}(\mathcal{Z})$ and $\text{D-mod}(\mathcal{Y})$ are compactly generated.

We have a pair of adjoint functors

$$i_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Z}) \rightleftarrows \text{D-mod}(\mathcal{Y}) : i^!.$$

Being a left adjoint, the functor $i_{\mathrm{dR},*}$ preserves compactness. But there is no reason for $i^!$ to have this property. We shall say that $\mathcal{Z}$ is *truncative* in $\mathcal{Y}$ if $i^!$ does preserve compactness.

Truncativeness is a purely "stacky" phenomenon. In Sect. 2.1.7 we will show that it never occurs for schemes, unless $\mathcal{Z}$ is a connected component of $\mathcal{Y}$.

Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be the embedding of the complementary open substack. We will say that U is *co-truncative* in $\mathcal{Y}$ if $\mathcal{Z}$ is truncative. This property can be reformulated as saying that the functor

$$j_* : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$$

preserves compactness. We show that the property of co-truncativeness can be also reformulated as existence of the functor $j_! : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$, *left* adjoint to the restriction functor j^*. (A priori, $j_!$ is only defined on the holonomic subcategory.)

Remark 0.2.2. The property of being compact for an object in $\text{D-mod}(\mathcal{Y})$ is somewhat subtle (e.g., it is not local in the smooth topology). In Sect. 2.1 we reformulate the notion of truncativeness and co-truncativeness in terms of the more accessible property of *coherence* instead of compactness.

0.2.3. Let us now drop the assumption that $\mathcal{Y}$ be quasi-compact. We shall say that a closed substack $\mathcal{Z}$ (resp., open substack U) is truncative (resp., co-truncative), if for every quasi-compact open $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}$, the intersection $\mathcal{Z} \cap \overset{\circ}{\mathcal{Y}}$ (resp., $U \cap \overset{\circ}{\mathcal{Y}}$) is truncative (resp., co-truncative) in $\overset{\circ}{\mathcal{Y}}$.

We shall say that $\mathcal{Y}$ is *truncatable* if it can be written as a union quasi-compact open substacks that are co-truncative.

Equivalently, $\mathcal{Y}$ is truncatable if the poset of quasi-compact open substacks of $\mathcal{Y}$ (with the order relation given by inclusion) contains a cofinal subset whose elements are co-truncative.

We will show (which is more or less tautological from [DrGa, Theorem 0.2.2]) that if $\mathcal{Y}$ is truncatable, then $\text{D-mod}(\mathcal{Y})$ is compactly generated.

0.2.4. Thus Theorem 0.1.2, is a consequence of the next statement, which is the main technical result of this paper:

Theorem 0.2.5. *Under the assumptions of Theorem 0.1.2, the stack* Bun_G *is truncatable.*

This theorem is proved in Sect. 6. The proof amounts to explicitly producing a family of quasi-compact open co-truncative substacks of Bun_G. Let us indicate what these substacks are.

0.2.6. In Sect. 5.3.6, to every dominant rational coweight θ we attach a quasi-compact open substack $\mathrm{Bun}_G^{(\leq\theta)} \subset \mathrm{Bun}_G$. Its k-points are those G-bundles $\mathcal{P}_G$ with the property that for every reduction $\mathcal{P}_B$ to the Borel, we have

$$\deg(\mathcal{P}_B) \underset{G}{\leq} \theta,$$

where $\deg(\mathcal{P}_B)$ is the degree of $\mathcal{P}_B$, which is a coweight of G, and $\underset{G}{\leq}$ is the usual partial ordering: $\lambda_1 \underset{G}{\leq} \lambda_2$ if $\lambda_2 - \lambda_1$ is a linear combination of simple coroots with non-negative coefficients.

Theorem 0.2.5 is a consequence of the following fact proved in Sect. 6:

Theorem 0.2.7. *The substack* $\mathrm{Bun}_G^{(\leq\theta)}$ *is co-truncative if for every simple root* $\check{\alpha}_i$ *one has*

$$\langle \theta \, , \check{\alpha}_i \rangle \geq 2g-2, \tag{0.1}$$

where g *is the genus of* X*.*

Condition (0.1) means that θ is "deep enough" inside the dominant chamber (of course, if $g \leq 1$ then the condition holds for any dominant θ).

0.3. Duality.

0.3.1. Recall the notion of dualizability of a DG category in the sense of Lurie (see Sect. 0.8.9). Any compactly generated DG category is automatically dualizable. In particular, such is $\text{D-mod}(\mathcal{Y})$ when $\mathcal{Y}$ is a truncatable algebraic stack.

0.3.2. However, more is true. As we recall in Sect. 1.3.9, if $\mathcal{Y}$ is quasi-compact, not only is the category $\text{D-mod}(\mathcal{Y})$ dualiable, but Verdier duality defines an equivalence

$$\text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y}).$$

It is natural to ask for a description of the dual category $\text{D-mod}(\mathcal{Y})$ when $\mathcal{Y}$ is no longer quasi-compact, but just truncatable.

0.3.3. As we shall see in Sect. 4.2, the category $\text{D-mod}(\mathcal{Y})^\vee$, denoted $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$, can be described explicitly, but it is a priori different from $\text{D-mod}(\mathcal{Y})$.

There exists a naturally defined functor $\mathrm{Id}_{\mathcal{Y}}^{\mathrm{naive}} : \text{D-mod}(\mathcal{Y})_{\mathrm{co}} \to \text{D-mod}(\mathcal{Y})$, but we show (see Proposition 4.3.5), that this functor is *not* an equivalence, unless the closure of every quasi-compact open in $\mathcal{Y}$ is again quasi-compact.

0.3.4. One can, however, define a less obvious functor $\text{D-mod}(\mathcal{Y})_{\mathrm{co}} \to \text{D-mod}(\mathcal{Y})$. This functor is not an equivalence in general. It differs from the identity functor even for $\mathcal{Y}$ quasi-compact, and does not have to be an equivalence in this case either (see Sect. 4.3.7 for more details).

However, this other functor turns out to be an equivalence if $\mathcal{Y} = \mathrm{Bun}_G$ with G reductive. In other words, the DG category $\text{D-mod}(\mathrm{Bun}_G)$ identifies with its dual in a non-trivial way and for non-trivial reasons. We shall address this in a subsequent publication.

0.4. Establishing truncativeness.

0.4.1. For the proof of Theorem 0.2.5, we will need to show that certain explicitly defined locally closed substacks of Bun_G are truncative. We shall do so by reducing our situation to a certain general paradigm that we call "the contraction principle", see Sect. 2.5.

Namely, we describe a general pattern $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ in which $\mathcal{Z}$ is contractive. We start with a schematic and affine map of stacks $\pi : \mathcal{W} \to \mathcal{X}$, equipped with section $\iota : \mathcal{X} \to \mathcal{W}$, and such that $\mathcal{W}$ is equipped with an action of $\mathbb{G}_m$ over $\mathcal{X}$, which *contracts* it onto the image of ι.

A prototypical example of this situation is when $\mathcal{X} = \mathrm{pt}$ and $\mathcal{W} = \mathbb{A}^n$, where ι corresponds to the 0 element, and $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by dilations.

We show that in this case

$$\mathcal{X}/\mathbb{G}_m \hookrightarrow \mathcal{W}/\mathbb{G}_m$$

is contrcative.

0.4.2. In fact, in Sect. 3 we generalize this contraction principle to a more general paradigm, which is expressed in more intrinsic with respect to $\mathcal{Z}$.

Namely, we give a definition of what it means for a locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ to be *contractive*. We prove Theorem 3.2.3 that says that if a locally closed substack is contractive, then it is truncative.

The property of being contractive is formulated as follows. We need there to exist a smooth cover $f : \mathcal{X} \to \mathcal{Z}$, which is acted on by automorphisms by the group $\mathbb{G}_m$, such that the f-pullback of $\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}}$ is strictly positively graded with respect to the $\mathbb{G}_m$-action on $f^*(\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}})$ arises from one on f.

0.4.3. In fact, we generalize the property of contractiveness even further, to one that we call "point-wise contractiveness", see Sect. 3.3.

Namely, we say that $\mathcal{Z}$ is point-wise contractive, if for every field-valued point $z \in \mathcal{Z}$ there exists a homomorphism $\mathbb{G}_m \to \mathrm{Aut}(z)$, which acts with strictly positive eigenvalues on the fiber of the (classical) conormal sheaf $\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}}$ at z, and with non-negative eigenvalues on the the fiber of the (classical) cotangent sheaf $T^*(\mathcal{Z})$ at z.

Point-wise contractiveness is strictly weaker than contractiveness. We state (but do not prove) the theorem that asserts that point-wise contractivess also implies truncativeness.

0.5. **Generalizations and open questions.** Let us return to the main result of the present paper, namely, Theorem 0.1.2.

0.5.1. In the situation of Quantum Geometric Langlands, one needs to consider the categories of twisted D-modules on Bun_G. The corresponding analog of Theorem 0.1.2, with the same proof, holds in this more general context.

0.5.2. Let $x_1, \ldots, x_n \in X$. Instead of Bun_G, consider the stack of G-bundles on X with a reduction to a parabolic P_i at x_i, $1 \leq i \leq n$. Most probably, in this situation an analog of Theorem 0.1.2 holds and can be proved in a similar way.

0.5.3. Suppose now that instead of reductions to parabolics (as in Subsect. 0.5.2), one considers deeper level structures at $x_1, \ldots, x_n$ (the simplest case being reduction to the unipotent radical of the Borel).

We do not know whether an analog of Theorem 0.1.2 holds in this case, and we do not know what to expect. In any case, our strategy of the proof of Theorem 0.1.2 fails in this context.

0.5.4. Here are some more questions:

Question 0.5.5. *Does the assertion of Theorem 0.1.2 (and its strengthening, Theorem 0.2.5) hold for* $\mathcal{Y}$ *being one of the stacks* $\overline{\mathrm{Bun}}_B$, Bun_P, $\overline{\mathrm{Bun}}_P$ *and* $\widetilde{\mathrm{Bun}}_P$, *where* B *is the Borel, and* P *a general parabolic?*

We are pretty confident that the answer is "yes" for $\overline{\mathrm{Bun}}_B$, but are less sure in other cases.

Question 0.5.6. *Does the assertion of Theorem 0.1.2 hold for an* arbitrary *connected affine algebraic group* G *(i.e., without the assumption that* G *be reductive)?*

0.6. **Organization of the paper.**

0.6.1. In Sect. 1 we review some general facts about the category of D-modules on an algebraic stack $\mathcal{Y}$.

We first consider the case when $\mathcal{Y}$ is quasi-compact and make a summary of the relevant results from [DrGa].

We then consider the case when $\mathcal{Y}$ is not quasi-compact and characterize the subcategory of D-mod($\mathcal{Y}$) formed by compact objects.

0.6.2. In Sect. 2, we introduce some of the main definitions for this paper: the notions of truncativeness (for a locally closed substack) and co-truncativeness (for an open substack).

We study the behavior of these notions under morphisms, base change, refinement of stratification, etc.

We also prove the "contraction principle", mentioned before that guarantees (in a very explicit way) that certain closed substacks are truncative.

0.6.3. In Sect. 3 we introduce a more general paradigm for establishing truncativeness, given by Theorem 3.2.3.

We prove that the geometric property of being contractive implies truncativeness.

We should mention that the proof of our the result, Theorem 0.1.2 can be deduced form the less general, but more geometric contraction principle of Sect. 2. So, the contents of Sect. 3 are not strictly speaking necessary if one only aims to prove Theorem 0.1.2.

0.6.4. In Sect. 4 we introduce the notion of truncatable stack.

We show that if $\mathcal{Y}$ is truncatable, then the category D-mod($\mathcal{Y}$) is compactly generated. In particular, we obtain that Theorem 0.2.5 implies Theorem 0.1.2.

We also discuss the behavior of Verdier duality on truncatable stacks, and the relation beween the category D-mod($\mathcal{Y}$) and its dual.

At the end of this section we prove Theorem 0.2.5 in the particular case of $G = SL_2$. The proof in the general case follows the same idea, but is more involved combinatorially.

0.6.5. In Sect. 5 we recall the Harder-Narasihman-Shatz stratification of Bun_G according to the degree of instability of the G-bundle.

We briefly indicate a way to establish the existence of such a stratification, using the relative compactification of the map $\mathrm{Bun}_P \to \mathrm{Bun}_G$.

0.6.6. In Sect. 6 we finally prove Theorem 0.2.5. The proof amounts to finding a way to combine the Harder-Narasihman-Shatz strata of Bun_G into locally closed substacks that are contractive, and hence truncative. In order to do it, we use the *Langlands retraction*[1] of the set $\Lambda^{\mathbb{Q}}$ of all rational coweights onto the dominant cone $\Lambda^{+,\mathbb{Q}}$ (several equivalent definitions of this retraction can be found in Subsection 6.1).

More precisely, we choose a dominant coweight η which is *deep enough* in the dominant chamber $\Lambda^{+,\mathbb{Q}}$ and consider the η-shifted Langands retraction

$$\Lambda^{+,\mathbb{Q}} \twoheadrightarrow (\eta + \Lambda^{+,\mathbb{Q}}). \tag{0.2}$$

The Harder-Narasihman-Shatz strata of Bun_G are labeled by elements of $\Lambda^{+,\mathbb{Q}}$, and the sought-for coarser strata correspond to the fibers of the map (0.2).

0.6.7. In Sect. 7 we show that it is *not* true that the category $\text{D-mod}(\mathcal{Y})$ is compactly generated for an arbitrary non quasi-compact stack $\mathcal{Y}$. Namely, we show that if $\mathcal{Y} = Y$ is a smooth scheme, which contains a non quasi-compact divisor, then the category $\text{D-mod}(\mathcal{Y})$ is not generated by compact objects.

More precisely, we show that (locally) coherent D-modules on Y that belong to the full category generated by compact objects cannot have all of $T^*(Y)$ as their singular support. In particular, the D-module $\mathcal{D}_Y$ does not belong to that subcategory.

0.6.8. In Appendix A we give an overview of the (well-known) material on the Langlands retraction. The appendix is self-contained and very elementary.

0.7. **Conventions and notation.**

0.7.1. Our conventions on ∞-categories follow those of [DrGa], listed in Sect. 0.6.1 of *loc.cit.*. Whenever we say "category", by default we mean an $(\infty, 1)$-category.

We denote by $\infty\text{-Cat}$ the $(\infty, 1)$-category of ∞-categories.

We denote by $\infty\text{-Grpd} \subset \infty\text{-Cat}$ in the $(\infty, 1)$-subcategory spanned by ∞-groupoids, a.k.a., spaces. We denote by $\mathbf{C} \mapsto \mathbf{C}^{\mathrm{grpd}}$ the functor $\infty\text{-Cat} \to \infty\text{-Grpd}$ right adjoint to the above embedding. Explicitly, $\mathbf{C}^{\mathrm{grpd}}$ is obtained from $\mathbf{C}$ by discarding non-invertible 1-morphisms.

For $\mathbf{C} \in \infty\text{-Cat}$ and objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$ we denote by $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{C}_2) \in \infty\text{-Grpd}$ the corresponding space of maps. We let $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ denote the set $\pi_0\,(\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{C}_2))$.

0.7.2. *Schemes and stacks.* This paper deals with categorical aspects of the category of D-modules. Therefore, all schemes and stacks in this paper can be considered *classical.* I.e., we do not need derived algebraic geometry for this paper.

However, all schemes, Artin stacks, stacks and prestacks in this paper are assumed *locally of finite type* over k. See [GL:Stacks], Sect. 1.3.2 (resp., Sect. 2.5) with $n = 0$ for the notion of classical prestack (resp., stack) locally of finite type. See *loc.cit.* Sect. 4.9 for the notion of classical Artin locally of finite type.

By an *algebraic stack* we shall mean one in the sense of [DrGa], Sect. 1.1.3.

[1]The fact that this retraction is relevant is not surprising in view of the historical remarks at the end of Appendix A.

0.7.3. *D-modules.* We refer the reader to the paper [GL:Crys] for the theory of D-modules (a.k.a. crystals) on prestacks locally of finite type.

For a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ of prestacks we have a tautologically defined functor

$$f^! : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1).$$

This functor may or may not have a left adjoint, which we denote by $f_!$.

If f is schematic and quasi-compact, we also have a functor of direct image

$$f_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2).$$

However, when f is an open embedding, we shall use the notation j_* instead of $j_{\mathrm{dR},*}$, and j^* instead of $j^!$, for reasons of tradition. This is not supposed to cause confusion, as the above functors go to the same-named functors for the underlying $\mathcal{O}$-modules.

0.7.4. *Ind-coherent sheaves.* In several places in this paper we shall also use the category $\mathrm{IndCoh}(\mathcal{Y})$ of ind-coherent sheaves and the forgetful functor

$$\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})} : \text{D-mod}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}).$$

We shall recall the relevant definitions when the need arises.

0.8. **DG categories: recollections and conventions.**

0.8.1. Throughout this paper we will work with DG categories. We refer the reader to [GL:DG] for a survey.

Our basic object of study is the $(\infty,1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$ whose objects are cocomplete DG categories (i.e., ones that contain arbitrary direct sums, or equivalently, colimits), and where 1-morphisms are continuous functors (i.e., exact functors that commute with arbitrary direct sums, or equivalently all colimits). [2]

The construction of $\mathrm{DGCat}_{\mathrm{cont}}$ as an $(\infty,1)$-category has not been fully documented. A pedantic reader can replace $\mathrm{DGCat}_{\mathrm{cont}}$ by an equivalent $(\infty,1)$-category of stable ∞-categories tensored over k, whose construction is a consequence of [Lu2, Sects. 4.2 and 6.3].

0.8.2. *Terminological deviation* (i). We shall sometimes encounter non-cocomplete DG categories (e.g., the subcategory of compact objects in a given DG category). Every time that this happens, we shall say so explicitly.

0.8.3. The category $\mathrm{DGCat}_{\mathrm{cont}}$ has a natural symmetric monoidal structure, given by Lurie's tensor product (see [Lu2], Sect. 6.1 or [GL:DG], Sect. 1.4 for a brief review).

Its unit object is the category Vect of chain complexes of k-vector spaces.

For $\mathbf{C}_1, \mathbf{C}_2 \in \mathrm{DGCat}_{\mathrm{cont}}$ we shall denote by $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_1, \mathbf{C}_2)$ their internal Hom in $\mathrm{DGCat}_{\mathrm{cont}}$, which is therefore another DG category.

[2] We shall ignore set-theoretic issues, however, the reader can assume that all DG categories are presentable.

0.8.4. *Terminological deviation* (ii). For two DG categories $\mathbf{C}_1$ and $\mathbf{C}_2$ we shall sometimes encounter functors $\mathbf{C}_1 \to \mathbf{C}_2$ that are not continuous (bus still exact). For example, for a non-compact object $\mathbf{c} \in \mathbf{C}$, such is the functor $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Vect}$ (see below for the notation).

Every time when we encounter a non-continuous functor, we shall say so explicitly.

All exact functors $\mathbf{C}_1 \to \mathbf{C}_2$ also form a DG category, which we denote by $\mathrm{Funct}(\mathbf{C}_1, \mathbf{C}_2)$. In fact, all DG categories with all functors between them form an $(\infty, 1)$-category that we denote DGCat. We have a forgetful functor $\mathrm{DGCat} \to \mathrm{DGCat}_{\mathrm{cont}}$ that induces an isomorphism on 2-morphisms and higher.

0.8.5. *Some notation.* Whenever a DG category $\mathbf{C}$ has a t-structure, we let $\mathbf{C}^{\leq 0}$ (resp., $\mathbf{C}^{\geq 0}$) denote the full subcategory of connective (resp., co-connective) objects. We denote by $\mathbf{C}^{\heartsuit}$ the heart of the t-structure.

Any DG category $\mathbf{C}$ can be thought of as an ∞-category enriched over Vect with the same set of objects. For two objects $\mathbf{c}_1, \mathbf{c}_2$, we shall denote by $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \mathrm{Vect}$ the corresponding Hom object.

Warning: This notation is different from that if [DrGa]. In *loc.cit.* we used $\mathrm{Hom}^{\bullet}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ instead of $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$.

We let $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \infty$-Grpd denote the Hom-space, when we consider $\mathbf{C}$ as a plain ∞-category. The object $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ equals the image of $\tau^{\leq 0}(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$ under the Dold-Kan functor

$$\mathrm{Vect}^{\leq 0} \to \infty\text{-Grpd}\,.$$

We denote by $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the object $H^0\left(\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)\right)$, which identifies with the set of homotopy classes of maps $\mathbf{c}_1 \to \mathbf{c}_2$ in $\mathbf{C}$ considered as a plain ∞-category.

0.8.6. *Compactness.* Recall that an object $\mathbf{c}$ in a (cocomplete) DG category $\mathbf{C}$ is called *compact* if the functor

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Vect}^{\heartsuit}$$

commutes with arbitrary direct sums. Thus is equivalent to the (a priori non-continuous) functor

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Vect}$$

being continuous.

For a DG category $\mathbf{C}$, we let $\mathbf{C}^c$ denotes the full (but not cocomplete) DG subcategory that consists of compact objects.

A DG category $\mathbf{C}$ is called compactly generated if there exists a set of compact objects $\mathbf{c}_\alpha$, such that

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_\alpha, \mathbf{c}) = 0, \ \forall \alpha \ \Rightarrow \mathbf{c} = 0. \tag{0.3}$$

Equivalently, $\mathbf{C}$ is compactly generated, if it does not contain proper full (cocomplete) DG subcategories that contain all of its compact objects.

0.8.7. The following observations will be used repeatedly throughout the paper:

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be a pair of DG categories, and let $\mathsf{G} : \mathbf{C}_2 \to \mathbf{C}_1$ be a (not necessarily continuous) functor. If G admits a left adjoint, then this left adjoint is automatically continuous.

Suppose now that $\mathbf{C}_1$ is compactly generated. Then G is continuous if and only if F sends compact objects of $\mathbf{C}_1$ to compact objects of $\mathbf{C}_2$.

0.8.8. Let $\mathbf{C}$ be a compactly generated category, and let $\mathbf{c}_\alpha \in \mathbf{C}$ be a set of compact objects.

Let $\mathbf{C}^0$ be the smallest full (but not cocomplete) DG subcategory of $\mathbf{C}^c$ that contains the objects $\mathbf{c}_\alpha$. (I.e., the objects of $\mathbf{C}^0$ are obtained from $\mathbf{c}_\alpha$'s by finite iterations of taking cones and finite direct sums.)

Suppose now that $\mathbf{c}_\alpha$'s generate $\mathbf{C}$. In this case, the the Thomason-Trobaugh-Neeman localization theorem (see [N, Theorem 2.1] or [BeV, Prop. 1.4.2]) says that the objects $\mathbf{c}_\alpha$ Karoubi-generate $\mathbf{C}^c$.

By definition, this means that every object of $\mathbf{C}^c$ is a direct summand of an object of $\mathbf{C}^0$.

0.8.9. *The notion of dual of a DG category.* A DG category $\mathbf{C}$ is called *dualizable* if it such as an object of $\text{DGCat}_{\text{cont}}$ considered as a symmetric monoidal category. We refer the reader to [DrGa], Sect. 4.1 for a review of some of the properties of this notion, which are relevant also to this paper. The most important ones are listed below:

For a dualizable category $\mathbf{C}$ we denote by $\mathbf{C}^\vee$ its dual. If $\mathsf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ is a (continuous) functor between dualizable categories, there exists a canonically defined dual functor $\mathsf{F}^\vee : \mathbf{C}_2^\vee \to \mathbf{C}_1^\vee$.

If $\mathbf{C}$ is compactly generated, then its dualizable. We have a canonical identification

$$(\mathbf{C}^\vee)^c \simeq (\mathbf{C}^c)^{op}.$$

Vice versa, if $\mathbf{C}_1$ and $\mathbf{C}_2$ are two compactly generated categories, then an identification

$$\mathbf{C}_1^c \simeq (\mathbf{C}_2^c)^{op}$$

gives rise to an identification

$$\mathbf{C}_1^\vee \simeq \mathbf{C}_2.$$

0.9. **Acknowledgements.** The research of V. D. is partially supported by NSF grant DMS-1001660. The research of D. G. is partially supported by NSF grant DMS-1063470.

1. Preliminaries on the DG category of D-modules on an algebraic stack

In this section we recall some definitions and results from [DrGa].

1.1. **Limits of DG categories.** The reason that we work with DG categories rather than with triangulated ones is that the limit (i.e., projective limit) of DG categories is well-defined as a DG category (while the corresponding fact for triangulated categories is false).

More precisely, the $(\infty, 1)$-categories $\text{DGCat}_{\text{cont}}$ and DGCat admit limits and the forgetful functor $\text{DGCat}_{\text{cont}} \to \text{DGCat}$ commutes with limits; see [GL:DG, Sect. 1.3]. This is important for us because the DG category of D-modules on an algebraic stack is defined as a limit (see Sect. 1.2.1 below).

1.1.1. Let
$$i \mapsto \mathbf{C}_i,\ (i \to j) \mapsto (\phi_{i,j} \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i, \mathbf{C}_j))$$
be a diagram of DG categories, parameterized by an index category I. The limit
$$\mathbf{C} := \underset{i \in I}{\underleftarrow{lim}}\, \mathbf{C}_i$$
is a priori defined by a universal property in $\mathrm{DGCat}_{\mathrm{cont}}$: for a DG category $\mathbf{D}$ we have a functorial isomorphism
$$(\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}))^{\mathrm{grpd}} \simeq \underset{i \in I}{\underleftarrow{lim}}\, (\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C}_i))^{\mathrm{grpd}}$$
where the in the left-hand side the limit is taken in the $(\infty, 1)$-category ∞-Grpd. We remind that the superscript "grpd" means that we are taking the maximal ∞-subgroupoid in the corresponding ∞-category.

1.1.2. Note that Corollary 3.3.3.2 from [Lu1] provides a more explicit description of $\mathbf{C}$. Namely, objects of $\mathbf{C}$ are *Cartesian sections,* i.e., assignments
$$i \mapsto (\mathbf{c}_i \in \mathbf{C}_i),\ \phi_{i,j}(\mathbf{c}_i) \overset{\alpha_{\phi_{i,j}}}{\simeq} \mathbf{c}_j,$$
equipped with data making $\alpha_{\phi_{i,j}}$ coherently associative. In fact, this description follows easily from the above functorial description, by taking $\mathbf{D} = \mathrm{Vect}$, and using the fact $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{Vect}, \mathbf{C}) \simeq \mathbf{C}$ as DG categories.

If $\mathbf{c} := (\mathbf{c}_i, \alpha_{\phi_{i,j}})$ and $\widetilde{\mathbf{c}} := (\widetilde{\mathbf{c}}_i, \widetilde{\alpha}_{\phi_{i,j}})$ are two such objects, then one can upgrade the assignment
$$i \mapsto \mathcal{M}aps_{\mathbf{C}_i}(\mathbf{c}_i, \widetilde{\mathbf{c}}_i)$$
into a homotopy I-diagram in Vect, and
$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, \widetilde{\mathbf{c}}) \simeq \underset{i \in I}{\underleftarrow{lim}}\, \mathcal{M}aps_{\mathbf{C}_i}(\mathbf{c}_i, \widetilde{\mathbf{c}}_i)$$
as objects of Vect.

1.1.3. The following observation will be useful in the sequel. Let $\mathbf{C} = \underset{i \in I}{\underleftarrow{lim}}\, \mathbf{C}_i$ be as above, and let
$$(\alpha \in A) \mapsto (\mathbf{c}_\alpha \in \mathbf{C})$$
be a collection of objects of $\mathbf{C}$ parameterized by some category A. In particular, for every $i \in I$ we obtain a functor
$$(\alpha \in A) \mapsto (\mathbf{c}_{i,\alpha} \in \mathbf{C}_i).$$

We have:

Lemma 1.1.4. *For every i, the map from $\underset{\alpha \in A}{\underrightarrow{colim}}\, \mathbf{c}_{i,\alpha} \in \mathbf{C}_i$ to the i-th component of the object $\underset{\alpha \in A}{\underrightarrow{colim}}\, \mathbf{c}_\alpha$ is an isomorphism.*

I.e., colimits in a limit of DG categories can be computed component-wise.

1.2. **D-modules on prestacks and algebraic stacks.**

1.2.1. Let $\mathcal{Y}$ be a prestack (always assumed locally of finite type). Recall following [DrGa] that the category D-mod($\mathcal{Y}$) is defined as the limit

$$\underset{S \in (\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}})^{op}}{\underset{\longleftarrow}{lim}} \text{D-mod}(S), \tag{1.1}$$

where the limit is taken in the $(\infty, 1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$. Here

$$S \mapsto \text{D-mod}(S)$$

is the functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{op} \to \mathrm{DGCat}_{\mathrm{cont}}$$

were for $f : S' \to S$ the corresponding map $\text{D-mod}(S) \to \text{D-mod}(S')$ is $f^!$.

I.e., as was explained in Sect. 1.1, informally, an object $\mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G)$ is an assignment for every $S \to \mathcal{Y}$ of an object $\mathcal{F}_S \in \text{D-mod}(S)$, and for every $f : S' \to S$ of an isomorphism $f^!(\mathcal{F}_S) \simeq \mathcal{F}_{S'}$.

In particular, for $\mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}(\mathrm{Bun}_G)$, the complex $\mathcal{M}aps(\mathcal{F}_1, \mathcal{F}_2)$ is calculated as

$$\underset{S \in (\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}})^{op}}{\underset{\longleftarrow}{lim}} \mathcal{M}aps_{\text{D-mod}(S)}((\mathcal{F}_1)_S, (\mathcal{F}_2)_S).$$

This definition has several variants. For example, we can replace the category of affine schemes by that of quasi-compact schemes, or all schemes.

1.2.2. Assume now that $\mathcal{Y}$ is an Artin stack (see [GL:Stacks], Sect. 4 for our conventions regarding Artin stacks).

In this case, as in [GL:IndCoh], Proposition 10.1.2, in the formation of the limit in (1.1), we can replace the category $\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$ by its non-full subcategory $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{smooth}})_{/\mathcal{Y}}$, where we restrict objects to be those pairs $(S, g : S \to \mathcal{Y})$ for which the map g is smooth, and 1-morphisms to smooth maps between affine schemes.

As before, we can replace the word "affine" by "quasi-compact", or just consider all schemes.

1.3. **D-modules on a quasi-compact algebraic stack.** Our primary interest in this paper is the category D-mod($\mathcal{Y}$) when $\mathcal{Y}$ is an algebraic stack in the sense of [DrGa], Sect. 1.1.3.

In this subsection, unless explicitly stated otherwise, we will assume that $\mathcal{Y}$ is quasi-compact (i.e., can be covered by an affine scheme). We shall impose a further condition on $\mathcal{Y}$, namely that the groups of automorphisms of its field-valued points are affine.

In [DrGa] we referred to the combination of the above two conditions (quasi-compactness and affinenness of automorphism groups of points) "QCA", see *loc.cit.*, Definition 1.1.8.

1.3.1. The following result is established in [DrGa], Theorem 7.1.1:

Theorem 1.3.2. *Under the above circumstances, the category* D-mod($\mathcal{Y}$) *is compactly generated.*

Remark 1.3.3. In fact, [DrGa, Theorem 7.1.1] produces an explicit set of compact generators of D-mod($\mathcal{Y}$). These are objects induced from coherent sheaves on $\mathcal{Y}$.

Remark 1.3.4. Before [DrGa], the above result was known for algebraic stacks that can be represented as Z/G, where Z is a quasi-compact scheme and G is an affine algebraic group acting on S. Most quasi-compact Artin stacks that appear in practice (e.g., all quasi-compact open substacks of Bun_G) admit such a representation. More generally, it was known for algebraic stacks that are perfect in the sense of [BFN].

1.3.5. *Compactness and coherence.* Let Z be a quasi-compact scheme. An object of $\text{D-mod}(Z)$ is said to be *coherent* if it is a bounded complex whose cohomology sheaves are coherent D-modules.

It is known that the (non cocomplete) subcategory $\text{D-mod}_{\mathrm{coh}}(Z)$ that consists of coherent objects coincides with $\text{D-mod}(Z)^c$. Recall from Sect. 0.8.6, that for a DG category $\mathbf{C}$ we denote by $\mathbf{C}^c$ the full subcategory spanned by compact objects.

For an algebraic stack $\mathcal{Y}$, an object $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is said to be *coherent* if $f^!(\mathcal{F})$ (or equivalently, $f^*_{\mathrm{dR}}(\mathcal{F})$) is coherent for any smooth map $f : Z \to \mathcal{Y}$, where Z is a quasi-compact scheme. So by definition, the property of coherence is local for the smooth topology. The full (but non-cocomplete) subcategory of coherent objects of $\text{D-mod}(\mathcal{Y})$ is denoted by $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$.

Theorem 1.3.6.
(i) *We have the inclusion* $\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$*.*
(ii) *The above inclusion is an equality if and only if for every geometric point y of $\mathcal{Y}$, the quotient of the automorphism group* $\mathrm{Aut}(y)$ *by its unipotent radical is finite.*

This theorem is proved in [DrGa, Corollary 9.2.7].

Remark 1.3.7. One may wonder how far is coherence from compactness. The answer is provided by the notion of *safety*, introduced in [DrGa, Sect. 8.2]. In Corollary 8.2.3 of *loc.cit.* it is shown that an object of $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ is compact if and only if it is safe.

Remark 1.3.8. Note that the notion of coherence of D-modules makes sense for *any* algebraic stack $\mathcal{Y}$, i.e., it does not have to be quasi-compact: we test it by smooth maps $Z \to \mathcal{Y}$, where Z is a quasi-compact scheme. The inclusion of point (i) of Theorem 1.3.6 remains valid in this context. The proof is very easy: for a map $f : Z \to \mathcal{Y}$, the functor f^*_{dR} sends compacts to compacts, because it admits a continuous right adjoint, namely $f_{\mathrm{dR},*}$.

1.3.9. *Verdier duality.* The (non-cocomplete) DG category $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ carries a natural anti-involution
$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : (\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}))^{op} \to \text{D-mod}_{\mathrm{coh}}(\mathcal{Y}),$$
which we refer to as *Verdier duality*, see [DrGa, Sect. 6.6.4].

The following key feature of this functor is established in *loc.cit.*, Corollary 7.3.2:

Theorem 1.3.10. *The functor* $\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}}$ *sends the subcategory*
$$(\text{D-mod}(\mathcal{Y})^c)^{op} \subset (\text{D-mod}_{\mathrm{coh}}(\mathcal{Y}))^{op}$$
to $\text{D-mod}(\mathcal{Y})^c \subset \text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$*.*

By Sect. 0.8.9, we obtain that the resulting functor
$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Y}} : (\text{D-mod}(\mathcal{Y})^c)^{op} \to \text{D-mod}(\mathcal{Y})^c$$
uniquely extends to an equivalence
$$\text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y}). \tag{1.2}$$

Alternatively, we can view the Verdier duality functor as follows. The DG category $\mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y}), \text{D-mod}(\mathcal{Y}))$ identifies tautologically with

$$\text{D-mod}(\mathcal{Y})^\vee \otimes \text{D-mod}(\mathcal{Y}).$$

The equivalence (1.2) is characterized by the property that the identity functor on D-mod($\mathcal{Y}$) corresponds to the object of $\text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(\mathcal{Y})$ equal to

$$(\Delta_{\mathcal{Y}})_{\mathrm{dR},*}(\omega_{\mathcal{Y}}),$$

where $\omega_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ is the dualizing object.

1.4. **Non quasi-compact algebraic stacks.** The main object of study of this paper is the stack Bun_G of principal G-bundles on a curve X, where G is a reductive group. The feature of Bun_G that will concern us in this paper is that it is not quasi-compact. In this subsection we shall make some preliminary comments about compact objects in the category of D-modules on such stacks.

1.4.1. Let $\mathcal{Y}$ be an algebraic stack. Throughout the paper we will be assuming that $\mathcal{Y}$ is locally QCA, i.e., that every quasi-compact open substack $U \subset \mathcal{Y}$ is QCA, so the category D-mod(U) is compactly generated by Theorem 1.3.2. However, it is not true, in general, that the category D-mod($\mathcal{Y}$) is compactly generated. For a counterexample, see Sect. 7.

By construction, the functor $\text{D-mod} : (\mathrm{PreStk}^{op}) \to \mathrm{DGCat}_{\mathrm{cont}}$ sends colimits in PreStk to limits in $\mathrm{DGCat}_{\mathrm{cont}}$. Hence, we obtain:

Lemma 1.4.2. *The restriction functor*

$$\text{D-mod}(\mathcal{Y}) \to \underset{U \subset \mathcal{Y}}{\underset{\longleftarrow}{lim}}\ \text{D-mod}(U),$$

is an equivalence, where the limit is taken over the poset of open quasi-compact substacks of $\mathcal{Y}$.

In particular, we obtain that for $\mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}(\mathcal{Y})$, the natural map

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}_1, \mathcal{F}_2) \to \underset{U \subset \mathcal{Y}}{\underset{\longleftarrow}{lim}}\ \mathcal{M}aps_{\text{D-mod}(U)}(\mathcal{F}_1|_U, \mathcal{F}_2|_U) \tag{1.3}$$

is an isomorphism.

The following observation will be useful in the sequel:

Corollary 1.4.3. *Suppose that a family of objects* $\mathcal{F}_\alpha \in \text{D-mod}(\mathcal{Y})$ *is locally finite, i.e., for every quasi-compact open* $U \subset \mathcal{Y}$ *the set of* α*'s such that* $\mathcal{F}_\alpha|_U \neq 0$ *is finite. Then the map*

$$\underset{\alpha}{\oplus} \mathcal{F}_\alpha \to \underset{\alpha}{\Pi} \mathcal{F}_\alpha$$

is an isomorphism.

Proof. Follows immediately from (1.3) and Lemma 1.1.4. □

1.4.4. Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be an open substack. We have a pair of (continuous) adjoint functors

$$j^* : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(U) : j_*.$$

In particular, the functor j^* sends $\text{D-mod}(\mathcal{Y})^c$ to $\text{D-mod}(U)^c$.

Now, the functor j^* has a *partially defined* left adjoint, denoted $j_!$. It again follows automatically that if for $\mathcal{F}_U \in \text{D-mod}(U)^c$, the object $j_!(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})$ is defined, then it is compact.

Since the functor j_* is fully faithful, we also obtain that whenever $j_!(\mathcal{F}_U)$ is defined, the canonical map

$$\mathcal{F}_U \to j^*(j_!(\mathcal{F}_U)) \tag{1.4}$$

is an isomorphism.

1.4.5. The next proposition gives a description of the subcategory $\text{D-mod}(\mathcal{Y})^c$:

Proposition 1.4.6. *An object $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is compact if and only if*

$$\mathcal{F} = j_!(\mathcal{F}_U) \tag{1.5}$$

for some open quasi-compact $U \overset{j}{\hookrightarrow} \mathcal{Y}$ and some $\mathcal{F}_U \in \text{D-mod}(U)^c$.

In formula (1.5) we mean that the partially defined functor $j_!$ is defined on $\mathcal{F}_U$, and the resulting object is isomorphic to $\mathcal{F}$.

By (1.4) we obtain that the object $\mathcal{F}_U$ can be recovered from $\mathcal{F}$ as $\mathcal{F}|_U := j^*(\mathcal{F})$.

1.4.7. Before proving Proposition 1.4.6, let us give two more reformulations of condition (1.5):

Lemma 1.4.8. *For $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ the following conditions are equivalent:*
(1) *$\mathcal{F} = j_!(\mathcal{F}_U)$ for some $\mathcal{F}_U \in \text{D-mod}(U)$.*
(2) *For any $\mathcal{F}_1 \in \text{D-mod}(\mathcal{Y})$, supported on $\mathcal{Y} - U$, we have $\mathrm{Hom}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}_1) = 0$.*
(3) *For any $U \overset{\tilde{j}}{\hookrightarrow} U' \overset{j'}{\hookrightarrow} \mathcal{Y}$, where U' is another open quasi-compact substack of $\mathcal{Y}$, we have:*

$$\mathcal{F}|_{U'} \simeq \tilde{j}_!(\mathcal{F}_U),$$

in particular, the object $\tilde{j}_!(\mathcal{F}_U)$ is defined.

Proof. By adjunction, (1)$\Leftrightarrow$(2). If (1) holds, it is clear by adjunction that $j_!(\mathcal{F}_U)|_{U_1}$ is canonically isomorphic to $\tilde{j}_!(\mathcal{F}_U)$, so (1)$\Rightarrow$(3).

Let us show that (3) implies (2). By formula (1.3), for any $\mathcal{F}, \mathcal{F}_1 \in \text{D-mod}(\mathcal{Y})$ one has

$$\mathcal{M}aps_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}_1) \simeq \underset{\underset{U'}{\longleftarrow}}{lim}\, \mathcal{H}om_{\text{D-mod}(U')}(\mathcal{F}|_{U'}, \mathcal{F}_1|_{U'}). \tag{1.6}$$

If $\mathcal{F}_1$ is supported on $\mathcal{Y} - U$ then all the terms in the RHS are zero, so the LHS is zero. $\square$

1.5. **Proof of Proposition 1.4.6.** As was remarked in Sect. 1.4.4, if (1.5) holds then compactness of $\mathcal{F}$ follows by adjunction.

Conversely, suppose $\mathcal{F} \in D(\mathcal{Y})$ is compact. Then by Sect. 1.4.4, for every open $U \subset \mathcal{Y}$ the object $\mathcal{F}|_U \in \text{D-mod}(U)$ is compact. So it remains to show that (1.5) holds for some quasi-compact open $U \overset{j}{\hookrightarrow} \mathcal{Y}$.

Assume the contrary. Using the equivalence (1)$\Leftrightarrow$(3) of Lemma 1.4.8, we obtain that for every quasi-compact open $U \subset \mathcal{Y}$ there is a quasi-compact open $U' \subset \mathcal{Y}$ containing U such that $(j_{U,U'})_!(\mathcal{F}|_U) \neq (\mathcal{F}|_{U'})$ (here $j_{U,U'} : U \hookrightarrow U'$).

Thus, we obtain an increasing sequence of open quasi-compact substacks $U_i \subset \mathcal{Y}$ such that $(j_{U_i,U_{i+1}})_!(\mathcal{F}|_{U_i}) \neq \mathcal{F}|_{U_{i+1}}$. Therefore, by Lemma 1.4.8, for each i there exists $\mathcal{E}_i \in \text{D-mod}(U_{i+1})$ such that $\mathcal{E}_i|_{U_i} = 0$ but $\mathrm{Hom}(\mathcal{F}|_{U_{i+1}}, \mathcal{E}_i) \neq 0$.

Let V be the union of the U_i's and let $\tilde{\mathcal{E}}_i \in \text{D-mod}(V)$ be the direct image of $\mathcal{E}_i$ under $U_i \hookrightarrow V$. Then

$$\mathrm{Hom}(\mathcal{F}|_V, \tilde{\mathcal{E}}_i) = \mathrm{Hom}(\mathcal{F}|_{U_{i+1}}, \mathcal{E}_i) \neq 0. \tag{1.7}$$

By Corollary 1.4.3,

$$\mathrm{Hom}(\mathcal{F}|_V, \underset{i}{\oplus}\, \tilde{\mathcal{E}}_i) \simeq \prod_i \mathrm{Hom}(\mathcal{F}|_V, \tilde{\mathcal{E}}_i). \tag{1.8}$$

On the other hand, by Sect. 1.4.4, $\mathcal{F}|_V$ is compact, so $\mathrm{Hom}(\mathcal{F}|_V, \underset{i}{\oplus}\, \tilde{\mathcal{E}}_i) \simeq \underset{i}{\oplus}\, \mathrm{Hom}(\mathcal{F}|_V, \tilde{\mathcal{E}}_i)$. This contradicts (1.8) because of (1.7). □

2. Truncativeness and co-truncativeness

In this section we let $\mathcal{Y}$ be a QCA algebraic stack.

2.1. **The notion of truncative substack.**

2.1.1. Let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be a closed substack, and let $\mathcal{Y} \overset{j}{\hookleftarrow} U$ be the complementary open. Consider the corresponding pairs of adjoint functors

$$i_{\mathrm{dR},*} : \text{D-mod}(\mathcal{Z}) \rightleftarrows \text{D-mod}(\mathcal{Y}) : i^! \text{ and } j^* : \text{D-mod}(\mathcal{Y}) \rightleftarrows \text{D-mod}(U) : j_*.$$

Recall that by Theorem 1.3.2, all the categories involved are compactly generated.

Proposition 2.1.2. *The following conditions are equivalent:*

(i) The functor $i^!$ sends D-mod$(\mathcal{Y})^c$ *to* D-mod$(\mathcal{Z})^c$*.*

(ii) The functor j_* sends D-mod$(U)^c$ *to* D-mod$(\mathcal{Y})^c$

(iii) The functor $j_!$, left adjoint to j^*, is defined on all of D-mod(U)*.*

(iii') The functor $j_!$, left adjoint to j^*, is defined on D-mod$(U)^c$*.*

(iv) The functor $i^!$ admits a continuos right adjoint.

Proof. Since j^* preserves compactness and $i_{\mathrm{dR},*}$ is fully faithful, the fact that (ii) implies (i) follows follows from the exact triangle

$$i_{\mathrm{dR},*}(i^!(\mathcal{F})) \to \mathcal{F} \to j_* \circ (j^*(\mathcal{F})).$$

Since the functor j_* is conservative, we obtain that the essential image of j^* generates D-mod(U). Hence, by Sect. 0.8.8, the essential image of D-mod$(X)^c$ under j^* Karoubi-generates D-mod$(U)^c$. Now, the fact that (i) implies (ii) follows from the same exact triangle.

The equivalence (i) $\Leftrightarrow$ (iv) is tautological.

Conditions (iii) and (iii') are equivalent since D-mod(U) is generated by D-mod$(U)^c$. It remains to establish the equivalence of (ii) and (iii').

Note that under the self-duality isomorphisms

$$\text{D-mod}(U)^\vee \simeq \text{D-mod}(U) \text{ and } \text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y}),$$

the dual of the functor j^* identifies with j_* (see [DrGa], Sect. 8.3.2 and Corollary 9.2.5, and use the fact that j is schematic, and hence safe).

Hence, the equivalence of (ii) and (iii') follows from the following (tautological) assertion (equivalent to [GL:DG, Lemma 2.3.3]):

Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a (continuous) functor between compactly generated DG categories, and consider the dual functor $F^\vee : \mathbf{C}_2^\vee \to \mathbf{C}_1^\vee$. Let $\mathbb{D}_{\mathbf{C}_1}$ denote the tautological equivalence $(\mathbf{C}_1^c)^{op} \to (\mathbf{C}_1^\vee)^c$.

Lemma 2.1.3. *For an object $\mathbf{c}_1 \in \mathbf{C}_1^c$, the partially defined left adjoint to $F^\vee$ is defined on $\mathbb{D}_{\mathbf{C}_1}(\mathbf{c}_1)$ if and only if $F(\mathbf{c}_1)$ is compact. In the latter case, the value of the above left adjoint is canonically isomorphic to $\mathbb{D}_{\mathbf{C}_2}(F(\mathbf{c}_1))$.*

□

Remark 2.1.4. Note that Lemma 2.1.3 gives an explicit way to calculate $j_!(\mathcal{F}_U)$ for $\mathcal{F}_U \in \text{D-mod}(U)^c$. Namely,
$$j_!(\mathcal{F}_U) \simeq \mathbb{D}^{\text{Verdier}}_{\mathcal{Y}}\left(j_*\left(\mathbb{D}^{\text{Verdier}}_U(\mathcal{F}_U)\right)\right).$$

2.1.5. We give the following definition crucial for this paper:

Definition 2.1.6. *A closed substack $\mathcal{Z} \subset \mathcal{Y}$ is called* truncative *(resp., an open substack $U \subset \mathcal{Y}$ is called* co-truncative*) if it satisfies the equivalent conditions of Proposition 2.1.2.*

2.1.7. Truncativeness a purely "stacky" phenomenon, i.e., it never happens for schemes.

More precisely, it is easy to see that if $j : U \hookrightarrow Y$ is an open embedding of schemes, which is not an inclusion of a union of connected components, then U cannot be truncative. Indeed, choose $\mathcal{M} \in \mathrm{Coh}(U)$ such that $j_*(\mathcal{M})$ is not coherent.

Then
$$j_*(\mathbf{ind}_{\text{D-mod}(U)}(\mathcal{M})) \simeq \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(j_*(\mathcal{M}))$$
is not in $\text{D-mod}(Y)^c$. Here $\mathbf{ind}_{\text{D-mod}(-)}$ denotes the induction functor from $\mathrm{IndCoh}(-)$ to $\text{D-mod}(-)$, see [DrGa, Sect. 5.1.3].

2.1.8. *Example.* Here is, however, the most basic example of a truncated/co-truncative substack:

Take $\mathcal{Y} = \mathbb{A}^n/\mathbb{G}_m$, where $\mathbb{G}_m$ acting on $\mathbb{A}^n$ by dilations. Take $U = (\mathbb{A}^n - \{0\})/\mathbb{G}_m \simeq \mathbb{P}^{n-1}$. In Sect. 2.5 we shall see that $U \hookrightarrow \mathcal{Y}$ is co-truncative.

2.1.9. The most basic case of the above example is when $n = 1$. In this case, the co-truncativeness assertion is particularly evident. Namely, let us see that condition (iii') of Proposition 2.1.2 is verified. Indeed, $\text{D-mod}(U) \simeq \mathrm{Vect}$, so it is sufficient to show that $j_!(k)$ is defined, where k is the generator of Vect. However, this is the case since we are dealing with holonomic D-modules.

2.1.10. Generalizing the example of Sect. 2.1.9 in a different direction, it is easy to see that when $\mathcal{Y}$ is such that it has only finitely many isomorphism classes of k-points, then any open substack is co-truncative. Indeed, we claim that condition (iii') of Proposition 2.1.2 is verified, because every object of $\text{D-mod}(U)^c$ is holonomic.

Examples of such $\mathcal{Y}$ include $N\backslash G/B$, or any quasi-compact open of Bun_G for X of genus 0.

2.1.11. *The non-standard functors.* Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be co-truncative. Since the functor j_* sends compact objects to compact ones, it admits a *continuous right adjoint*. We denote this right adjoint by $j^?$. We emphasize that $j^?$ is *not* one of the standard functors considered in the theory of D-modules.

Similarly, if $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is a truncative closed substack, we denote the right adjoint to $i^!$ by $i_?$.

For example, in the situation of Sect. 2.1.9, it is easy to see that the functor $i_?$ equals $(p/\mathbb{G}_m)^!$, where $(p/\mathbb{G}_m) : \mathbb{A}^1/\mathbb{G}_m \to \mathrm{pt}/\mathbb{G}_m$.

2.2. **Truncativeness and coherence.** As was mentioned in Sect. 1.3.5, the property of compactness of a D-module on a stack is subtle. For example, it is not local in the smooth topology. We are going to reformulate the notion of truncativeness via the more accessible property, namely, coherence.

2.2.1. Let
$$\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y} \overset{j}{\hookleftarrow} U$$
be as above. We claim:

Proposition 2.2.2.

(a) $\mathcal{Z}$ *is truncative if and only of the functor* $i^!$ *sends* $\text{D-mod}_{\text{coh}}(\mathcal{Y})$ *to* $\text{D-mod}_{\text{coh}}(\mathcal{Z})$.

(b) U *is co-truncative if and only if* j_* *sends* $\text{D-mod}_{\text{coh}}(U)$ *to* $\text{D-mod}_{\text{coh}}(\mathcal{Y})$.

Proof. To prove the "if" implications in both (a) and (b) we shall use the notion of *safety* from [DrGa, Sect. 8.2], and the fact that for a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ between QCA stacks, the functor $f_{\text{dR},*}$ always preserves safety, and $f^!$ preserves safety if f itself is safe (in particular, when f is schematic); see *loc. cit.*, Lemma 9.4.2.

Thus, the "if" implications follow from the fact that "compatness=coherence+safety", see [DrGa, Proposition 8.2.3].

To prove the "only if" implication in (a) we argue as follows.

Lemma 2.2.3. *For a QCA stack* $\mathcal{Y}$*, an object* $\mathcal{F} \in \text{D-mod}_{\text{coh}}(\mathcal{Y})$ *and an integer* n*, there exists* $\mathcal{F}' \in \text{D-mod}(\mathcal{Y})^c$ *and a map* $\mathcal{F}' \to \mathcal{F}$*, such that its cone lies in* $\text{D-mod}(\mathcal{Y})^{\leq -n}$.

Proof. This follows from the fact that the abelian category $\text{D-mod}(\mathcal{Y})^\heartsuit$ is Noetherian, and that every object of
$$\text{D-mod}_{\text{coh}}(\mathcal{Y})^\heartsuit := \text{D-mod}(\mathcal{Y})^\heartsuit \cap \text{D-mod}_{\text{coh}}(\mathcal{Y})$$
receives a surjection from a compact object. Compact objects in question can be taken of the form $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{M})$ for $\mathcal{M} \in \text{Coh}(\mathcal{Y})$. □

Note that the functor $i^!$ is left t-exact, and has a finite cohomological amplitude, say k. For $\mathcal{F} \in \text{D-mod}_{\text{coh}}(\mathcal{Y})$, which lies in $\text{D-mod}(\mathcal{Y})^{\geq -m}$, choose $\mathcal{F}'$ as in Lemma 2.2.3 with $n > k + m$. Consider the exact triangle
$$i^!(\mathcal{F}') \to i^!(\mathcal{F}) \to i^!(\mathcal{F}''),$$
where $\mathcal{F}'' := \text{Cone}(\mathcal{F}' \to \mathcal{F})$. By construction, the maps
$$\tau^{\geq -m}(i^!(\mathcal{F}')) \to \tau^{\geq -m}(i^!(\mathcal{F})) \to i^!(\mathcal{F}) \tag{2.1}$$
are isomorphisms.

By assumption, $i^!(\mathcal{F}') \in \text{D-mod}(\mathcal{Z})^c \subset \text{D-mod}_{\text{coh}}(\mathcal{Z})$. Note also that the truncation functors preserve the subcategory $\text{D-mod}_{\text{coh}}(-)$. Hence $\tau^{>-m}(i^!(\mathcal{F}')) \in \text{D-mod}_{\text{coh}}(\mathcal{Z})$. Hence, (2.1) implies that $i^!(\mathcal{F}) \in \text{D-mod}_{\text{coh}}(\mathcal{Z})$, as desired.

The "only if" implication in (b) is proved similarly.

□

2.3. **Stability of truncativeness.** In this subsection we will establish some basic properties of the truncativeness/co-truncativeness condition. Let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y} \overset{j}{\hookleftarrow} U$ be as above.

2.3.1. First, we have:

Lemma 2.3.2. *Let $\mathcal{Z} \hookrightarrow \mathcal{Y}$ be truncative. Then for any QCA stack $\mathcal{X}$, the closed embedding*
$$\mathcal{Z} \times \mathcal{X} \hookrightarrow \mathcal{Y} \times \mathcal{X}$$
is also truncative.

Proof. By [DrGa, Corollary 7.3.4], for a pair of QCA stacks $\mathcal{X}_1$ and $\mathcal{X}_2$, the natural functor
$$\text{D-mod}(\mathcal{X}_1) \otimes \text{D-mod}(\mathcal{X}_2) \to \text{D-mod}(\mathcal{X}_1 \times \mathcal{X}_2)$$
is an equivalence.

We obtain that the functor
$$(j \times \mathrm{id}_{\mathcal{X}})_! : \text{D-mod}(U \times \mathcal{X}) \to \text{D-mod}(\mathcal{Y} \times \mathcal{X}),$$
left adjoint to $(j \times \mathrm{id}_{\mathcal{X}})^*$, is given by $j_! \otimes \mathrm{Id}_{\text{D-mod}(\mathcal{X})}$.

□

2.3.3. Assume now that there exists a smooth surjective morphism $f : \widetilde{\mathcal{Y}} \to \mathcal{Y}$, such that the resulting substack
$$\widetilde{\mathcal{Z}} := \mathcal{Z} \underset{\mathcal{Y}}{\times} \widetilde{\mathcal{Y}} \overset{\widetilde{i}}{\hookrightarrow} \widetilde{\mathcal{Y}}$$
is truncative.

Proposition 2.3.4. *Under the above circumstances $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is truncative.*

Proof. We shall use Proposition 2.2.2(a). It is sufficient to show that the functor $i^!$ sends $\text{D-mod}_{\mathrm{coh}}(\mathcal{Y})$ to $\text{D-mod}_{\mathrm{coh}}(\mathcal{Z})$.

Let f' denote the resulting morphism $\widetilde{\mathcal{Z}} \to \mathcal{Z}$. Since f' is smooth and surjective, it is easy to see that for $\mathcal{F}' \in \text{D-mod}(\mathcal{Z})$,
$$\mathcal{F}' \in \text{D-mod}_{\mathrm{coh}}(\mathcal{Z}) \Leftrightarrow f'^!(\mathcal{F}') \in \text{D-mod}_{\mathrm{coh}}(\widetilde{\mathcal{Z}}).$$

However,
$$f'^! \circ i^! \simeq \widetilde{i}^! \circ f^!,$$
and the assertion follows.

□

Remark 2.3.5. Note that the converse to Proposition 2.3.4 is false: truncativeness downstairs does *not* imply truncativeness upstairs. E.g., consider the example of $\mathrm{pt}/\mathbb{G}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m$ smoothly covered by $\mathrm{pt} \hookrightarrow \mathbb{A}^1$.

However, the converse to Proposition 2.3.4 does hold for étale schematic morphisms:

Lemma 2.3.6. *Suppose that $f : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ is étale and schematic. If $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is truncative, then so is $\widetilde{\mathcal{Z}} \hookrightarrow \widetilde{\mathcal{Y}}$.*

Proof. The functor $f_{\mathrm{dR},*} : \text{D-mod}(\widetilde{\mathcal{Y}}) \to \text{D-mod}(\mathcal{Y})$ is conservative. Hence, by Sect. 0.8.8, the essential image of $\text{D-mod}(\mathcal{Y})^c$ under $f^*_{\mathrm{dR}} \simeq f^!$ Karoubi-generates $\text{D-mod}(\widetilde{\mathcal{Y}})^c$. Hence, it is enough to show that $\widetilde{i}^! \circ f^!$ preserves compactness. However,
$$\widetilde{i}^! \circ f^! \simeq f'^! \circ i^!.$$
Now, $i^!$ preserves compactness by assumption, and $f'^!$ preserves compactness, because it is isomorphic to f'^*_{dR}, which is the left adjoint of a continuous functor, namely, $f'_{\mathrm{dR},*}$.

□

2.3.7. Note also, if $f : \widetilde{\mathcal{Y}} \hookrightarrow \mathcal{Y}$ is a closed embedding, then if $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is schematic, then so is

$$\widetilde{\mathcal{Z}} := \mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Y}' \hookrightarrow \mathcal{Y}'.$$

(This follows from the fact that an object $\mathcal{F} \in \text{D-mod}(\widetilde{\mathcal{Y}})$ is compact if and only if $f_{\mathrm{dR},*}(\mathcal{F}) \in \text{D-mod}(\mathcal{Y})$ is, because the functor $f_{\mathrm{dR},*}$ is fully faithful.)

The above claim admits the following generalization:

Proposition 2.3.8. *Let $f : \widetilde{\mathcal{Y}} \hookrightarrow \mathcal{Y}$ be a finite schematic morphism. If $\mathcal{Z}$ is truncative in $\mathcal{Y}$, then $\widetilde{\mathcal{Z}}$ is truncative in $\widetilde{\mathcal{Y}}$.*

Proof. We need to show that the functor $\widetilde{i}^!$ preserves coherence. Since f is finite, the functor $f_{\mathrm{dR},*}$ preserves coherence. Hence, we obtain that the functor

$$i^! \circ f_{\mathrm{dR},*} \simeq f'_{\mathrm{dR},*} \circ \widetilde{i}^!$$

preserves coherence. Now, the assertion of the proposition follows from the next general lemma:

Lemma 2.3.9. *Let $g : \mathcal{X}' \to \mathcal{X}$ be a finite schematic map. If $\mathcal{F}' \in \text{D-mod}(\mathcal{X})$ is such that $g_{\mathrm{dR},*}(\mathcal{F}') \in \text{D-mod}(\mathcal{X})$ is coherent, then $\mathcal{F}'$ is coherent.*

□

Proof. (of Lemma 2.3.9) Follows immediately from the fact that the functor $g_{\mathrm{dR},*}$ is t-exact and conservative. □

Remark 2.3.10. One can combine Lemma 2.3.6 and Proposition 2.3.8 to the following assertion: the assertion of Proposition 2.3.8 continues to hold when f is a quasi-finite compactifiable morphism.

2.3.11. Finally, we have the following. Let

$$\begin{CD}
\widetilde{\mathcal{Z}} @>{\widetilde{i}}>> \widetilde{\mathcal{Y}} \\
@V{f'}VV @VV{f}V \\
\mathcal{Z} @>{i}>> \mathcal{Y}
\end{CD}$$

be a Cartesian diagram with f schematic, proper and surjective, and i a closed embedding.

Lemma 2.3.12. *If under the above circumstances $\widetilde{\mathcal{Z}}$ os truncative in $\widetilde{\mathcal{Y}}$, then $\mathcal{Z}$ is truncative in $\mathcal{Y}$.*

Proof. First, as in [GL:IndCoh, Proposition 4.5.3], the functor $f^!$ is conservative. Hence, the essential image of $f_{\mathrm{dR},*}$ generates $\text{D-mod}(\mathcal{Y})$. Hence, by Sect. 0.8.8, the essential image of $\text{D-mod}(\widetilde{\mathcal{Y}})^c$ under $f_{\mathrm{dR},*}$ Karoubi-generates $\text{D-mod}(\mathcal{Y})^c$. Therefore, it is sufficient to show that the functor $i^! \circ f_{\mathrm{dR},*}$ preserves compactness. However, by base change

$$i^! \circ f_{\mathrm{dR},*} \simeq f'_{\mathrm{dR},*} \circ \widetilde{i}^!,$$

while $\widetilde{i}^!$ preserves compactness by assumption, and $f'_{\mathrm{dR},*}$ preserves compactness by properness. □

2.4. **Truncativeness of locally closed substacks.**

2.4.1. Let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ be a locally closed substack, i.e., the corresponding map of stacks becomes a locally closed embedding after a base change $Z \to \mathcal{Y}$, where Z is a scheme. This condition is enough to verify for just one smooth or flat covering $Z \to \mathcal{Y}$.

As in the case of schemes, it is easy to see that every locally closed embedding can be factored (and even canonically so) as

$$\mathcal{Z} \overset{i'}{\hookrightarrow} \mathcal{Y}' \overset{j}{\hookrightarrow} \mathcal{Y},$$

where i' is a closed embedding, and j is an open embedding. Indeed, define $\mathcal{Z}'$ is a closure of $\mathcal{Z}$ in $\mathcal{Y}$, so that $\mathcal{Z}$ is open in $\mathcal{Z}$, and let $\mathcal{Y}' := \mathcal{Y} - (\mathcal{Z}' - \mathcal{Z})$.

Definition 2.4.2. *We shall say that $\mathcal{Z}$ is truncative if the functor $i^!$ preserves compactness.*

We have:

Lemma 2.4.3. *The substack is truncative if and only if for some/any open substack $\mathcal{Y}' \subset \mathcal{Y}$, such that i factors as*

$$\mathcal{Z} \overset{i'}{\hookrightarrow} \mathcal{Y}' \overset{j}{\hookrightarrow} \mathcal{Y},$$

with i' being a closed embedding, $\mathcal{Z}$ in truncative in $\mathcal{Y}'$.

Proof. The "if" direction is tautological. The "only if" direction follows from the fact that the essential image of $\text{D-mod}(\mathcal{Y})^c$ under j^* Karoubi-generates $\text{D-mod}(\mathcal{Y}')$. $\square$

The above lemma implies:

Corollary 2.4.4. *$\mathcal{Z}$ is truncative inside $\mathcal{Y}$ if and only if the functor $i^!$ preserves coherence.*

Corollary 2.4.5. *The property of $\mathcal{Z}$ being truncative in $\mathcal{Y}$ is stable under base change by locally closed embeddings $\widetilde{\mathcal{Y}} \to \mathcal{Y}$.*

2.4.6. From the definitions we obtain:

Lemma 2.4.7. *Let $\mathcal{Y}_1 \hookrightarrow \mathcal{Y}_2 \hookrightarrow \mathcal{Y}_3$ be locally closed embeddings. If $\mathcal{Y}_1$ is truncative in $\mathcal{Y}_2$ and $\mathcal{Y}_2$ is truncative in $\mathcal{Y}_3$, then $\mathcal{Y}_1$ is truncative in $\mathcal{Y}_3$.*

Corollary 2.4.8. *If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are locally closed truncative substacks of $\mathcal{Y}$, then so is $\mathcal{Z}_1 \cap \mathcal{Z}_2$.*

Proof. By Corollary 2.4.5, $\mathcal{Z}_1 \cap \mathcal{Z}_2$ is truncative in $\mathcal{Z}_1$. Now, the assertion follows from Lemma 2.4.7. $\square$

2.4.9. Finally, we have the following assertion:

Proposition 2.4.10. *Let $\mathcal{Z}$ be a locally closed substack of $\mathcal{Y}$, equal to the union of locally closed substacks $\mathcal{Z}_i$, $i = 0, 1, ..., n$. If each $\mathcal{Z}_i$ is truncative in $\mathcal{Y}$, then so is $\mathcal{Z}$.*

Proof. By induction, the assertion reduces to the case when $n = 1$ with $\mathcal{Z}_0$ open in $\mathcal{Z}$, and $\mathcal{Z}_1 = \mathcal{Z} - \mathcal{Z}_0$. Furthermore, by Lemma 2.4.3, with no restriction of generality, we can assume that $\mathcal{Z}$ is closed in $\mathcal{Y}$. Hence, we find ourselves in the following situation:

Let $\mathcal{Z}_1 \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{Y}$ be closed embeddings. Denote $\mathcal{Z}_0 := \mathcal{Z} - \mathcal{Z}_1 \hookrightarrow \mathcal{Y}$. Assume that $\mathcal{Z}_1$ and $\mathcal{Z}_0$ are truncative in $\mathcal{Y}$. We need to show that so is $\mathcal{Z}$.

Consider the open substacks $\mathcal{Y} - \mathcal{Z} \subset \mathcal{Y} - \mathcal{Z}_1 \subset \mathcal{Y}$. The fact that $\mathcal{Z}_1$ is truncative in $\mathcal{Y}$ means by definition that $\mathcal{Y} - \mathcal{Z}_1$ is co-truncative in $\mathcal{Y}$. The fact that $\mathcal{Z}_0$ is truncative in $\mathcal{Y}$ implies that $\mathcal{Z}_0$ is truncative in $\mathcal{Y} - \mathcal{Z}_1$, i.e., that $\mathcal{Y} - \mathcal{Z}$ is co-truncative in $\mathcal{Y} - \mathcal{Z}_1$. Since the relation of co-truncativeness is transitive, we obtain that $\mathcal{Y} - \mathcal{Z}$ is co-truncative in $\mathcal{Y}$, as required. $\square$

2.5. **The contraction principle.** We are going to explain a general paridigm that would guarantee that a certain locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is truncative.

We are going to do it in two iterations: the first one, considered in this subsection is more geometric but less general.

The second one, considered in Sect. 3, generalizes the first one, but the proof goes beyond the framework of the category D-mod$(-)$ and the standard functors; it will use the fact that we can induce D-modules from $\mathcal{O}$-modules.

2.5.1. Consider the following set-up. Let $\pi : \mathcal{W} \to \mathcal{X}$ be a schematic affine map between QCA algebraic stacks, equipped with a section $\iota : \mathcal{X} \to \mathcal{W}$. Thus, ι realizes $\mathcal{X}$ as a closed substack of $\mathcal{W}$.

We assume that $\mathcal{W}$ is acted on by the monoid $\mathbb{A}^1$ (with respect to the operation of multiplication on $\mathbb{A}^1$), compatible with the projection to $\mathcal{X}$ (so that the action of $\mathbb{A}^1$ on $\mathcal{X}$) is trivial. Assume that the action of $\{0\} \in \mathbb{A}^1$ is the endomorphism of $\mathcal{W}$ equal to

$$\mathcal{W} \xrightarrow{\pi} \mathcal{X} \xrightarrow{\iota} \mathcal{W}.$$

I.e., informally, we can say that the action of $\mathbb{G}_m \subset \mathbb{A}^1$ "contracts" $\mathcal{W}$ onto the image of $\mathcal{X}$ under i.

Set $\mathcal{Y} := \mathcal{W}/\mathbb{G}_m$ and $\mathcal{Z} := \mathcal{X}/\mathbb{G}_m$, where $\mathbb{G}_m$ acts trivially on $\mathcal{X}$. Let i denote the resulting closed embedding $\mathcal{Z} \hookrightarrow \mathcal{Y}$. We claim:

Proposition 2.5.2. *Under the above circumstances, the closed substack $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{Y}$ is truncative.*

The rest of this subsection is devoted to the proof of Proposition 2.5.2. First, we note that the statement reduces tautologically to the case when $\mathcal{X}$ (and hence also $\mathcal{W}$) are quasi-compact schemes (or even affine schemes). We will change the notation and denote $X := \mathcal{X}$ and $W := \mathcal{W}$.

2.5.3. We obtain that W identifies with $\operatorname{Spec}_X(\mathcal{A})$, where $\mathcal{A}$ is a quasi-coherent sheaf of non-negatively graded $\mathcal{O}_X$-algebras with $\mathcal{A}_0 \simeq \mathcal{O}_X$. The section ι corresponds to the projection on the 0-component

$$\mathcal{A} \to \mathcal{A}_0 \simeq \mathcal{O}_X.$$

For an integer n, let $\mathcal{A}^{(n)} \subset \mathcal{A}$ be the subalgebra $\underset{i}{\oplus} \mathcal{A}_{i\cdot n}$. It is easy to see that the map

$$\operatorname{Spec}_X(\mathcal{A}) \to \operatorname{Spec}_X(\mathcal{A}^{(n)})$$

is finite.

Consider Cartesian diagram

$$\begin{tikzcd}
X \underset{\operatorname{Spec}_X(\mathcal{A}^{(n)})}{\times} \operatorname{Spec}_X(\mathcal{A}) \arrow[r] \arrow[d] & \operatorname{Spec}_X(\mathcal{A}) \arrow[d] \\
X \arrow[r] & \operatorname{Spec}_X(\mathcal{A}^{(n)}).
\end{tikzcd}$$

The map $\iota : X \to \operatorname{Spec}_X(\mathcal{A})$ factors as

$$X \to X \underset{\operatorname{Spec}_X(\mathcal{A}^{(n)})}{\times} \operatorname{Spec}_X(\mathcal{A}) \to \operatorname{Spec}_X(\mathcal{A}),$$

where the first arrow induces an isomorphism at the level of reduced schemes.

By Proposition 2.3.8, in the statement of the proposition we can therefore replace $\mathcal{A}$ by $\mathcal{A}^{(n)}$. It is easy to see that for n large enough, the algebra $\mathcal{A}^{(n)}$ is locally generated by sections of degree n, which we shall from now on assume.

2.5.4. Replacing $\mathcal{A}$ by $\mathcal{A}^{(n)}$, we obtain that the action of $\mathbb{G}_m$ on W factors through the map

$$\mathbb{G}_m \xrightarrow{x\mapsto x^n} \mathbb{G}_m, \tag{2.2}$$

and we have a Cartesian diagram

$$\begin{tikzcd} X/\mathbb{G}_m \arrow[r, "i"] \arrow[d] & W/\mathbb{G}_m \arrow[d] \\ X/\mathbb{G}_m \arrow[r] & W/\mathbb{G}_m, \end{tikzcd}$$

where in the bottom row we use the action "divided by n", and the vertical arrows are given by (2.2).

Since the vertical arrows in the above diagram are finite, by Proposition 2.3.8 we can therefore assume that $n = 1$, and that $\mathcal{A}$ is locally generated by sections of degree 1. Thus, we can assume that W admits a $\mathbb{G}_m$-equivariant closed embedding into the total space of a vector bundle E over X, where $\mathbb{G}_m$ acts on E by multiplication.

By Proposition 2.3.8 (for closed embeddings) we can replace W by E, while ι corresponds to the 0-section of E. In this case we shall prove the assertion of Proposition 2.5.2 explicitly.

2.5.5. We have the following well-known assertion (this is a particular case of Braden's theorem [Br]):

Lemma 2.5.6. *The functor*

$$(\pi/\mathbb{G}_m)_! : \text{D-mod}(E/\mathbb{G}_m) \to \text{D-mod}(X/\mathbb{G}_m) = \text{D-mod}(X \times \mathrm{pt}/\mathbb{G}_m)$$

is well-defined, and is canonically isomorphic to $i^!$.

We will supply a proof for completeness (see below).

Let us show how Lemma 2.5.6 implies Proposition 2.5.2. We will verify condition (iv) in Proposition 2.1.2. We need to show that $i^!$ admits a continuous right adjoint. However, by Lemma 2.5.6, the right adjoint in question is given by

$$\text{D-mod}(X \times \mathrm{pt}/\mathbb{G}_m) = \text{D-mod}(X/\mathbb{G}_m) \xrightarrow{(\pi/\mathbb{G}_m)^!} \text{D-mod}(E/\mathbb{G}_m).$$

$\square$(Proposition 2.5.2)

Remark 2.5.7. Note that we have just calculated the functor

$$i_? : \text{D-mod}(X/\mathbb{G}_m) \to \text{D-mod}(E/\mathbb{G}_m).$$

Namely, we have shown that it identifies with $(\pi/\mathbb{G}_m)^!$.

2.5.8. *Proof of Lemma 2.5.6.* For $\mathcal{F} \in \text{D-mod}(E/\mathbb{G}_m)$, consider the exact triangle

$$i_{\mathrm{dR},*} \circ i^!(\mathcal{F}) \to \mathcal{F} \to j_* \circ j^*(\mathcal{F}).$$

We claim that for any $\mathcal{F}' \in \text{D-mod}(E - X/\mathbb{G}_m) = \text{D-mod}(\mathbb{P}(E))$, we have $(\pi/\mathbb{G}_m)_!(j_*(\mathcal{F}')) = 0$.

We will show that for any $\mathcal{F}' \in \text{D-mod}(E - X)$ that lies in the essential image of the pullback functor

$$\text{D-mod}(\mathbb{P}(E)) \to \text{D-mod}(E - X),$$

we have $\pi_!(\jmath_*(\mathcal{F}')) = 0 \in \text{D-mod}(X)$, where $\jmath : E - X \hookrightarrow E$. This will imply the assertion of the lemma since the pullback functor $\text{D-mod}(X/\mathbb{G}_m) = \text{D-mod}(X \times \mathrm{pt}/\mathbb{G}_m) \to \text{D-mod}(X)$ is conservative.

Let $\widetilde{E} \stackrel{q}{\to} E$ be the blow-up of E along the 0-section. Let $\widetilde{\pi}$ denote the resulting projection $\widetilde{E} \to \mathbb{P}(E)$. Let $\widetilde{\jmath}$ denote the open embedding $E - X \hookrightarrow \widetilde{E}$.

Since the map q is proper, we have:

$$\pi_!(\jmath_*(\mathcal{F}')) \simeq \pi_!(q_* \circ \widetilde{\jmath}_*(\mathcal{F}')) \simeq (\pi \circ q)_!(\widetilde{\jmath}_*(\mathcal{F}')),$$

so we need to show that $(\pi \circ q)_!(\widetilde{\jmath}_*(\mathcal{F}')) = 0$.

Note that the map

$$\widetilde{E} \xrightarrow{q} E \xrightarrow{p} X$$

equals

$$\widetilde{E} \xrightarrow{\widetilde{\pi}} \mathbb{P}(E) \to X.$$

Hence, it suffices to show that $(\widetilde{\pi})_!(\widetilde{\jmath}_*(\mathcal{F}')) = 0$. This reduces the assertion to the case when E is a line bundle, since such is $\widetilde{E}$ over $\mathbb{P}(E)$.

The assertion is Zariski-local, so we can assume that E is constant. This reduces the assertion to the case when $X = \mathrm{pt}$.

Thus, it remains to calculate $H_{\mathrm{dR},c}(\mathbb{A}^1, \jmath_*(k))$, where $\jmath : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$. However, the latter is known to be zero.

$\square$(Lemma 2.5.6)

3. An "intrinsic" contraction principle

In this section all stacks are assumed to be QCA,

3.1. An "intrinsic" action of $\mathbb{G}_m$.

3.1.1. Let $\mathcal{Z}$ be a stack. Let $\mathcal{X}$ be another stack, and let $f : \mathcal{X} \to \mathcal{Z}$ be a map. Assume that we are given an action of $\mathbb{G}_m$ by automorphisms of f.

This data can be reformulated as follows: consider the intertia stack $\mathrm{Inert}(\mathcal{Z})$ of $\mathcal{Z}$, i.e., $\mathcal{Z} \underset{\mathcal{Z}\times\mathcal{Z}}{\times} \mathcal{Z}$, and let us view it as a group-scheme over $\mathcal{Z}$. The data of action of $\mathbb{G}_m$ on f is equivalent to that of homomorphism

$$\mathbb{G}_m \times \mathcal{X} \to \mathcal{X} \underset{\mathcal{Z}}{\times} \mathrm{Inert}(\mathcal{Z}).$$

3.1.2. *Example.* Let us take $\mathcal{X} = \mathcal{Z}$ and $f = \mathrm{id}$, so we obtain a $\mathbb{G}_m$ action on the identity map of $\mathcal{Z}$. This data can be equivalently reformulated as an action of the group-stack $(\mathrm{pt}/\mathbb{G}_m)$ on $\mathcal{Z}$.

3.1.3. Returning to the general case of $f : \mathcal{X} \to \mathcal{Z}$, we obtain that the morphism f can be canonically factored as

$$\mathcal{X} \to \mathrm{pt}/\mathbb{G}_m \times \mathcal{X} \to \mathcal{Z}.$$

Hence, for every $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$, the pullback $f^*(\mathcal{F}) \in \mathrm{QCoh}(\mathcal{X})$ is naturally $\mathbb{G}_m$-equivariant with respect to the *trivial* $\mathbb{G}_m$-action on $\mathcal{X}$. In other words, $f^*(\mathcal{F})$ acquires a canonical $\mathbb{Z}$-grading.

We shall say that for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$ its weights with respect to $\mathbb{G}_m$ belong to a subset $\mathcal{S} \subset \mathbb{Z}$ if this is the case for $f^*(\mathcal{F})$. This notion depends of course on the choice of f and the $\mathbb{G}_m$-action on it.

A similar discussion applies to $\mathrm{IndCoh}(-)$.

3.1.4. Recall the induction functor

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Z})} : \mathrm{IndCoh}(\mathcal{Z}) \to \text{D-mod}(\mathcal{Z})$$

of [DrGa], Sect. 6.3, left adjoint to the forgetful functor

$$\mathbf{oblv}_{\text{D-mod}(\mathcal{Z})} : \text{D-mod}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z}).$$

We have:

Proposition 3.1.5. *Assume that f is a smooth schematic cover. Then there exists a finite subset $\mathcal{S} \subset \mathbb{Z}$ such that for any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Z})$ whose weights are outside of $\mathcal{S}$, we have $\mathbf{ind}_{\text{D-mod}(\mathcal{Z})}(\mathcal{F}) = 0$.*

The proof will be given in Sect. 3.5.

3.2. **The notion of contractive substack.**

3.2.1. Let $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ be a locally closed embedding. Let $\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}} \in \mathrm{Coh}(\mathcal{Z})$ be the (non-derived) conormal sheaf, i.e., $\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}} = \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the sheaf of ideals corresponding to $\mathcal{Z}$ inside $\mathcal{Y}$.

Definition 3.2.2. *A locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is said to be* contractive *if there exists a smooth schematic cover $f : \mathcal{X} \to \mathcal{Z}$ and an action of $\mathbb{G}_m$ on f by automorphisms such that the $\mathbb{G}_m$-weights of $\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}}$ are strictly positive.*

We will prove:

Theorem 3.2.3. *If a locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is contractive, then it is truncative.*

Remark 3.2.4. Note that the situation of Proposition 2.5.2 falls into the paradigm of Theorem 3.2.3. So, the former is a particular case of the latter.

3.3. **Point-wise contractiveness.**

3.3.1. In addition to the notion of contractiveness, we can define yet another notion:

Definition 3.3.2. *We shall say that a locally closed substack $\mathcal{Z} \subset \mathcal{Y}$ is* point-wise contractive *if every field-valued point $z \in \mathcal{Z}$ admits a homomorphism $\mathbb{G}_m \to \mathrm{Aut}(z)$, such that the eigenvalues of its action on $(\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}})_z$ are strictly positive, and the eigenvalues of its action on all of $(T^*(\mathcal{Z}))_z$ are non-negative, where $T^*(\mathcal{Z})$ denotes the* classical *cotangent sheaf of $\mathcal{Z}$.*

It is easy to see that if $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is contractive, then it is point-wise contractive.

In addition, the property of point-wise contractiveness is much easier to verify than that of contractiveness.

3.3.3. We have the following result:

Theorem 3.3.4. *If a locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is point-wise contractive, then it is truncative.*

The proof will be supplied elsewhere. It combines the idea of the proof of Theorem 3.2.3 given below with those of [DrGa, Theorem 1.4.2].

Remark 3.3.5. We note that truncativeness is still much weaker than point-wise contractiveness. E.g., in the situation of Sect. 2.1.10, any locally closed subset is truncative, but it does not at all have to be point-wise contractive.

3.4. **Proof of Theorem 3.2.3.**

3.4.1. With no restriction of generality, we can assume that $\mathcal{Z}$ is closed in $\mathcal{Y}$. Let

$$\mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Z}} \subset \mathrm{IndCoh}(\mathcal{Y})$$

be the full subcategory of objects with set-theoretic support on $\mathcal{Z}$. Let

$$'i_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{Y})_{\mathcal{Z}} \rightleftarrows \mathrm{IndCoh}(\mathcal{Y}) : {}'i^!.$$

For $n \geq 0$, let $\mathcal{Z}_n \overset{i_n}{\hookrightarrow} \mathcal{Y}$ denote the (classical) n-th infinitesimal neighborhood of $\mathcal{Z}$ in $\mathcal{Y}$. Let $(i_n)_*^{\mathrm{IndCoh}} \circ i_n^!$ denote the resulting endo-functor of $\mathrm{IndCoh}(\mathcal{Y})$. We have a functorial isomorphism

$$'i_*^{\mathrm{IndCoh}} \circ {}'i^! \simeq \underset{n \in \mathbb{Z}^{\geq 0}}{\underset{\longrightarrow}{colim}}\, \mathsf{F}_n.$$

Let $\mathcal{I}_n$ be the sheaf of ideals of $\mathcal{Z}_n$. Consider the object

$$\mathcal{I}_n/\mathcal{I}_{n+1} \in \mathrm{Coh}(\mathcal{Z})^{\heartsuit}.$$

3.4.2. Let us recall the following construction. For two objects $\mathcal{F}', \mathcal{F}'' \in \mathrm{IndCoh}(\mathcal{Z})$ with $\mathcal{F}' \in \mathrm{Coh}(\mathcal{Z})$ we can consider their *internal Hom* object

$$\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{F}', \mathcal{F}'') \in \mathrm{QCoh}(\mathcal{Z}),$$

see [GL:DG],Sect. 5.1. It is characterized by the property that for $\mathcal{G} \in \mathrm{QCoh}(\mathcal{Z})$, we have

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{G}, \underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{F}', \mathcal{F}'')) = \mathrm{Hom}_{\mathrm{IndCoh}(\mathcal{Z})}(\mathcal{G} \underset{\mathcal{O}_{\mathcal{Z}}}{\otimes} \mathcal{F}', \mathcal{F}''),$$

where $- \underset{\mathcal{O}_{\mathcal{Z}}}{\otimes} -$ denote the canonical action of $\mathrm{QCoh}(\mathcal{Z})$ on $\mathrm{IndCoh}(\mathcal{Z})$.

Note that if $\mathcal{F}'' \in \mathrm{IndCoh}(\mathcal{Z})^+$, then $\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{F}', \mathcal{F}'') \in \mathrm{QCoh}(\mathcal{Z})^+$, so we can view it as an object of $\mathrm{IndCoh}(\mathcal{Z})$ via the equivalence

$$\Psi_{\mathcal{Z}} : \mathrm{IndCoh}(\mathcal{Z})^+ \to \mathrm{QCoh}(\mathcal{Z})^+$$

(see [GL:IndCoh, Corollary 10.2.6]).

3.4.3. We shall deduce Theorem 3.2.3 from the following (doubtless, well-known) assertion:

Lemma 3.4.4. *For $n \geq 0$ and $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Z})^+$, the cone $\mathrm{Cone}(\mathsf{F}_{n-1}(\mathcal{F}) \to \mathsf{F}_n(\mathcal{F}))$ is canonically isomorphic to*

$$i_*^{\mathrm{IndCoh}} \left(\underline{\mathrm{Hom}}_{\mathrm{QCoh}(\mathcal{Z})}(\mathcal{I}_n/\mathcal{I}_{n+1}, i^!(\mathcal{F})) \right).$$

Proof. As both sides belong to $\mathrm{IndCoh}(\mathcal{Y})^+$, it suffices to prove the isomorphism after applying the functor $\Psi_{\mathcal{Y}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})$. I.e., we need to show that for $\mathcal{G} \in \mathrm{QCoh}(\mathcal{Y})$, there is a canonical isomorphism

$$\mathrm{Cone}\left(\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}((i_{n-1})_* \circ i_{n-1}^*(\mathcal{G}), \mathcal{F}) \to \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})}((i_n)_* \circ i_n^*(\mathcal{G}), \mathcal{F}) \right) \simeq$$
$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})} \left(i_* \circ i^*(\mathcal{G} \underset{\mathcal{O}_{\mathcal{Z}}}{\otimes} \mathcal{I}_n/\mathcal{I}_{n+1}), \mathcal{F} \right).$$

We rewrite the left-hand side as

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})} \left(\mathcal{G} \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} \mathrm{Cone}\left((i_n)_*(\mathcal{O}_{\mathcal{Z}_n}) \to (i_{n-1})_*(\mathcal{O}_{\mathcal{Z}_{n-1}})\right), \mathcal{F} \right),$$

and the right-hand side as

$$\mathcal{M}aps_{\mathrm{QCoh}(\mathcal{Y})} \left(\mathcal{G} \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} i_*(\mathcal{I}_n/\mathcal{I}_{n+1}), \mathcal{F} \right),$$

and the two are evidently isomorphic.

□

3.4.5. Let us return to the proof of Theorem 3.2.3. We need to show that the functor

$$i^! : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Z})$$

preserves compactness, which is equivalent to the fact that $i_{\mathrm{dR},*} \circ i^!$ does.

Since $\mathcal{Y}$ is QCA, by [DrGa, Corollary 7.3.1], the category $\text{D-mod}(\mathcal{Y})^c$ is Karoubi-generated by objects of the form $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F})$ with $\mathcal{F} \in \mathrm{Coh}(\mathcal{Y})^\heartsuit$. Hence, it is enough to show that the functor

$$i_{\mathrm{dR},*} \circ i^! \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})} : \mathrm{IndCoh}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y})$$

sends $\mathrm{Coh}(\mathcal{Y})^\heartsuit$ to $\text{D-mod}(\mathcal{Y})^c$.

Let $j : U \hookrightarrow \mathcal{Y}$ be the complementary open embedding. We have:

$$'i_*^{\mathrm{IndCoh}} \circ {'i^!} \simeq \mathrm{Cone}(\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{Y})} \to j_*^{\mathrm{IndCoh}} \circ j^{\mathrm{IndCoh},*})[-1]$$

and

$$i_{\mathrm{dR},*} \circ i^! \simeq \mathrm{Cone}(\mathrm{Id}_{\text{D-mod}(\mathcal{Y})} \to j_* \circ j^*)[-1].$$

Hence,

$$i_{\mathrm{dR},*} \circ i^! \circ \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}(\mathcal{F}) \simeq \mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ {'i_*^{\mathrm{IndCoh}}} \circ {'i^!}.$$

Note that the functor $'i_*^{\mathrm{IndCoh}} \circ {'i^!}$ is of bounded cohomological amplitude (because j_*^{IndCoh} is). The functor $\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}$ is right t-exact, being the left adjoint of $\mathbf{oblv}_{\text{D-mod}(\mathcal{Y})}$, which is left t-exact.

Hence, for $\mathcal{F} \in \mathrm{Coh}(\mathcal{Y})^\heartsuit$, the object

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ {'i_*^{\mathrm{IndCoh}}} \circ {'i^!}(\mathcal{F}) \in \text{D-mod}(\mathcal{Y})$$

is bounded above. To prove the proposition, it is enough to show that its individual cohomologies

$$H^k\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ {'i_*^{\mathrm{IndCoh}}} \circ {'i^!}(\mathcal{F})\right)$$

are all compact as objects of $\text{D-mod}(\mathcal{Y})$.

3.4.6. We write $'i_*^{\mathrm{IndCoh}} \circ {'i^!}(\mathcal{F})$ as

$$\underset{n \in \mathbb{Z}^{\geq 0}}{\underrightarrow{colim}}\, \mathsf{F}_n(\mathcal{F}),$$

and hence

$$H^k\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ {'i_*^{\mathrm{IndCoh}}} \circ {'i^!}(\mathcal{F})\right) \simeq \underset{n \in \mathbb{Z}^{\geq 0}}{\underrightarrow{colim}}\, H^k\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ \mathsf{F}_n(\mathcal{F})\right).$$

We claim now that for a given $\mathcal{F} \in \mathrm{Coh}(\mathcal{Y})^\heartsuit$ and $k \in \mathbb{Z}$, we have:

(I) Each $H^k\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ \mathsf{F}_n(\mathcal{F})\right)$ is compact.

(II) For $n \gg 0$, the maps

$$H^k\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ \mathsf{F}_{n-1}(\mathcal{F})\right) \to H^k\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})} \circ \mathsf{F}_n(\mathcal{F})\right)$$

are isomorphisms.

The combination of (I) and (II) implies the assertion of Theorem 3.2.3.

3.4.7. To prove (I), we write

$$H^k\left(\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\circ \mathsf{F}_n(\mathcal{F})\right)\simeq \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\left(H^k(\mathsf{F}_n(\mathcal{F}))\right)\simeq \mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\circ (i_n)_*^{\text{IndCoh}}\left(H^k(i_n^!(\mathcal{F}))\right).$$

However, it is clear that $H^k(i_n^!(\mathcal{F}))$ belongs to $\text{Coh}(\mathcal{Z}_n)$, and hence

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\circ (i_n)_*^{\text{IndCoh}}\left(H^k(i_n^!(\mathcal{F}))\right)\in \text{D-mod}(\mathcal{Y})^c.$$

3.4.8. To prove (II), by the long exact sequence, we need to show that

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\left(H^k\left(\text{Cone}(\mathsf{F}_{n-1}(\mathcal{F})\to \mathsf{F}_n(\mathcal{F}))\right)\right)=0$$

for a fixed k and $n\gg 0$.

By Lemma 3.4.4, we rewrite the latter expression as

$$\mathbf{ind}_{\text{D-mod}(\mathcal{Y})}\circ i_*^{\text{IndCoh}}\left(H^k\left(\underline{\text{Hom}}_{\text{QCoh}(\mathcal{Z})}(\mathcal{I}_n/\mathcal{I}_{n+1}, i^!(\mathcal{F}))\right)\right),$$

and further as

$$i_{\text{dR},*}\circ \mathbf{ind}_{\text{D-mod}(\mathcal{Z})}\left(H^k\left(\underline{\text{Hom}}_{\text{QCoh}(\mathcal{Z})}(\mathcal{I}_n/\mathcal{I}_{n+1}, i^!(\mathcal{F}))\right)\right).$$

By Proposition 3.1.5, it suffices to show that for $n\gg 0$, the weights of

$$H^k\left(\underline{\text{Hom}}_{\text{QCoh}(\mathcal{Z})}(\mathcal{I}_n/\mathcal{I}_{n+1}, i^!(\mathcal{F}))\right),$$

with respect to the $\mathbb{G}_m$ action are outside of the finite subset $\mathcal{S}\subset\mathbb{Z}$ given by Proposition 3.1.5.

3.4.9. The map $\tau^{\leq k}(i^!(\mathcal{F}))\to i^(\mathcal{F})$ induces an isomorphism

$$H^k\left(\underline{\text{Hom}}_{\text{QCoh}(\mathcal{Z})}\left(\mathcal{I}_n/\mathcal{I}_{n+1}, \tau^{\leq k}(i^!(\mathcal{F}))\right)\right)\to H^k\left(\underline{\text{Hom}}_{\text{QCoh}(\mathcal{Z})}(\mathcal{I}_n/\mathcal{I}_{n+1}, i^!(\mathcal{F}))\right).$$

Note that $\tau^{\leq k}(i^!(\mathcal{F}))$ is coherent. Hence, there exists an integer m such that the weights of $\tau^{\leq k}(i^!(\mathcal{F}))$ are $<m$.

Note also that we have a surjection of coherent sheaves on $\mathcal{Z}$

$$\text{Sym}^n_{\mathcal{O}_\mathcal{Z}}(\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}})\twoheadrightarrow \mathcal{I}_n/\mathcal{I}_{n+1}.$$

Now, the assumption on $\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}}$ implies that the weights on $\text{Sym}^n_{\mathcal{O}_\mathcal{Z}}(\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}})$ are $\geq n$. Hence, the same is true for $\mathcal{I}_n/\mathcal{I}_{n+1}$.

Now, taking $n>m-\min(\mathcal{S})$ for m as above, we obtain that the weights of

$$\underline{\text{Hom}}_{\text{QCoh}(\mathcal{Z})}\left(\mathcal{I}_n/\mathcal{I}_{n+1}, \tau^{\leq k}(i^!(\mathcal{F}))\right)$$

are outside of $\mathcal{S}$ as desired.

$\square$(Theorem 3.2.3)

3.5. **Proof of Proposition 3.1.5.**

3.5.1. Consider the relative tangent complex $T(\mathcal{X}/\mathcal{Z})\in\text{Coh}(\mathcal{X})$. Since f is smooth and scematic, $T(\mathcal{X}/\mathcal{Z})$ is a vector bundle on $\mathcal{X}$.

The $\mathbb{G}_m$-action on f endows $T(\mathcal{X}/\mathcal{Z})$ with a grading. We let $\mathcal{S}$ be the set of weights that appear in the exterior powers of $T(\mathcal{X}/\mathcal{Z})$. We need to show that this choice of $\mathcal{S}$ satisfies the condition of the proposition.

3.5.2. Let $\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}$ be the category of relative D-modules on $\mathcal{X}$ (see [DrGa, Sect. 6.3.1]). Let $(\mathbf{ind}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}}, \mathbf{oblv}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}})$ and $(\mathbf{ind}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}}, \mathbf{oblv}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}})$ denote the resulting pairs of adjoint functors

$$\mathbf{ind}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}} : \text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}} \rightleftarrows \text{D-mod}(\mathcal{X}) : \mathbf{oblv}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}}$$

and

$$\mathbf{ind}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}} : \mathrm{IndCoh}(\mathcal{X}) \rightleftarrows \text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}} : \mathbf{oblv}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}},$$

respectively.

We have the commutative diagram,

$$\begin{tikzcd}
\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}} \arrow[d, "f_{\mathrm{dR}_{\mathrm{rel}_{\mathcal{Z}}},*}"'] & \text{D-mod}(\mathcal{X}) \arrow[l, "\mathbf{oblv}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}}"'] \arrow[d, "f_{\mathrm{dR},*}"] \\
\mathrm{IndCoh}(\mathcal{Z}) & \text{D-mod}(\mathcal{Z}) \arrow[l, "\mathbf{oblv}_{\text{D-mod}(\mathcal{Z})}"']
\end{tikzcd}$$

and by adjunction

$$\begin{tikzcd}
\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}} \arrow[r, "\mathbf{ind}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}}"] & \text{D-mod}(\mathcal{X}) \\
\mathrm{IndCoh}(\mathcal{Z}) \arrow[u, "f^*_{\mathrm{dR}_{\mathrm{rel}_{\mathcal{Z}}}}"] \arrow[r, "\mathbf{ind}_{\text{D-mod}(\mathcal{Z})}"] & \text{D-mod}(\mathcal{Z}), \arrow[u, "f^*_{\mathrm{dR}}"']
\end{tikzcd}$$

where f^*_{dR} and $f^*_{\mathrm{dR}_{\mathrm{rel}_{\mathcal{Z}}}}$ are the left adjoints of $f_{\mathrm{dR},*}$ and $f_{\mathrm{dR}_{\mathrm{rel}_{\mathcal{Z}}},*}$, respectively, which are well-defined because f is smooth.

In addition, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathcal{Z}) & \text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}} \arrow[l, "\mathbf{oblv}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}}"'] & \text{D-mod}(\mathcal{X}) \arrow[l, "\mathbf{oblv}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}}"'] \\
\mathrm{IndCoh}(\mathcal{Z}) \arrow[u, "f^!"] & \mathrm{IndCoh}(\mathcal{Z}) \arrow[l, "\mathrm{Id}"'] \arrow[u, "f^!"'] & \text{D-mod}(\mathcal{Z}). \arrow[l, "\mathbf{oblv}_{\text{D-mod}(\mathcal{Z})}"'] \arrow[u, "f^!"']
\end{tikzcd}$$

The functors

$$f^!, f^*_{\mathrm{dR}_{\mathrm{rel}_{\mathcal{Z}}}} : \mathrm{IndCoh}(\mathcal{Z}) \rightrightarrows \text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}$$

satisfy

$$f^*_{\mathrm{dR}_{\mathrm{rel}_{\mathcal{Z}}}} \simeq f^![-2\dim(\mathcal{X}/\mathcal{Z})].$$

3.5.3. Let $\mathcal{F}$ be an object of $\mathrm{IndCoh}(\mathcal{X})$, whose weights are outside of $\mathcal{S}$. It suffices to show that

$$f^*_{\mathrm{dR}}(\mathbf{ind}_{\text{D-mod}(\mathcal{Z})}(\mathcal{F})) = 0,$$

which is equivalent to

$$\mathbf{ind}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}\to\mathrm{abs}}}(f^!(\mathcal{F})) = 0,$$

i.e., that

$$\mathrm{Hom}_{\text{D-mod}(\mathcal{X})_{\mathrm{rel}_{\mathcal{Z}}}}(f^!(\mathcal{F}), \mathcal{M}) = 0$$

for any D-module $\mathcal{M}$ on $\mathcal{X}$.

The $\mathbb{G}_m$-action on f defines a map of Lie algebroids

$$\mathcal{O}_{\mathcal{X}} \otimes \mathrm{Lie}(\mathbb{G}_m) \to T(\mathcal{X}/\mathcal{Z}),$$

where $\mathcal{O}_\mathcal{X} \otimes \text{Lie}(\mathbb{G}_m)$ is considered as a trivial algebroid on $\mathcal{X}$. In particular, $\mathcal{O}_\mathcal{X} \otimes \text{Lie}(\mathbb{G}_m)$ is a quasi-coherent sheaf of Lie algebras on $\mathcal{X}$, and for $\mathcal{M} \in \text{D-mod}(\mathcal{X})$, the induced $\mathcal{O}_\mathcal{X} \otimes \text{Lie}(\mathbb{G}_m)$-action on $\mathcal{M}$ is zero.

Note that the weights of action of $\mathcal{O}_\mathcal{X} \otimes \text{Lie}(\mathbb{G}_m)$ on $f^!(\mathcal{F})$ are obtained from the weights of $\mathcal{F}$ by adding the weight of $\Lambda^{\text{top}}(T^*(\mathcal{X}/\mathcal{Z}))$. Now, the required assertion follows from the following general observation:

3.5.4. Let $\mathcal{T}$ be a Lie algebroid on $\mathcal{X}$, which receives a homomorphism from a trivial algebroid $\mathcal{O}_\mathcal{X} \otimes \text{Lie}(\mathbb{G}_m)$. We assume that the adjoint action of $\mathcal{O}_\mathcal{X} \otimes \text{Lie}(\mathbb{G}_m)$ on $\mathcal{T}$ comes from a $\mathbb{Z}$-grading.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two right $\mathcal{T}$-modules, such that the induced action of $\mathcal{O}_\mathcal{X} \otimes \text{Lie}(\mathbb{G}_m)$ on $\mathcal{M}_1$ and $\mathcal{M}_2$ comes from a $\mathbb{Z}$-grading. Assume that the set $\text{wt}(\mathcal{M}_2)$ of weights of $\mathcal{M}_2$ does not intersect the set

$$\text{wt}(\mathcal{M}_1) + \underset{i=0,\dots,\text{rk}(\mathcal{T})}{\cup} \text{wt}(\Lambda^i(\mathcal{T})).$$

Lemma 3.5.5. *Under the above circumstances,* $\text{Hom}_{\mathcal{T}\text{-mod}}(\mathcal{M}_1, \mathcal{M}_2) = 0$.

Proof. Follows from the de Rham complex that computes $\text{Hom}_{\mathcal{T}\text{-mod}}(\mathcal{M}_1, \mathcal{M}_2)$. □

□(Proposition 3.1.5)

4. Truncatable stacks

For the rest of the paper we shall take $\mathcal{Y}$ to be an algebraic stack, which is locally QCA.

In this section we will formulate a condition on $\mathcal{Y}$, called "truncatibility", which would guarantee that the category $\text{D-mod}(\mathcal{Y})$ is compactly generated.

4.1. The notion of truncatibility.

4.1.1. First, let us adapt the notions of truncativeness and co-truncativeness to the non quasi-compact situation:

Definition 4.1.2. *A locally closed substack $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is said to be truncative, if for every open quasi-compact substack $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}$, the intersection $\mathcal{Z} \cap \overset{\circ}{\mathcal{Y}}$ is truncative in $\overset{\circ}{\mathcal{Y}}$.*

Similarly,

Definition 4.1.3. *An open substack $U \overset{j}{\hookrightarrow} \mathcal{Y}$ is said to be co-truncative, if for every open quasi-compact substack $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}$, the intersection $U \cap \overset{\circ}{\mathcal{Y}}$ is co-truncative in $\overset{\circ}{\mathcal{Y}}$.*

Note that it follows from Corollary 2.4.8 that if U_1 and U_2 are co-truncative, then so is $U_1 \cup U_2$.

4.1.4. As in Lemma 1.4.8, it is easy to see that U is co-truncative if and only if the functor $j_!$, left adjoint to j^*, is defined.

In addition, by Sect. 2.1.11, if U is co-truncative, the functor $j_* : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$ admits a *continuous right adjoint*, that we denote by $j^?$.

4.1.5. We are now ready to give the following definition, crucial for this paper:

Definition 4.1.6. *The stack $\mathcal{Y}$ is said to be truncatable if it it can be covered by open quasi-compact substacks that are co-truncative.*

Equivalently, we can rephrase this definition as follows:

Lemma 4.1.7. *$\mathcal{Y}$ is truncatable if and only if every open quasi-compact substack is contained in one which is co-truncative. Equivalently, $\mathcal{Y}$ is truncatable if and only if the sub-posed of co-truncative open quasi-compact substacks in $\mathcal{Y}$ is cofinal among all open quasi-compact substacks.*

Corollary 4.1.8. *If $\mathcal{Y}$ is truncatable, then the natural restriction functor*

$$\text{D-mod}(\mathcal{Y}) \to \underset{U\subset\mathcal{Y}}{\underleftarrow{lim}}\ \text{D-mod}(U),$$

is an equivalence, where the limit is taken over the poset of co-truncative open quasi-compact substacks of $\mathcal{Y}$.

4.1.9. We claim:

Proposition 4.1.10. *If $\mathcal{Y}$ is truncatable, then the category* $\text{D-mod}(\mathcal{Y})$ *is compactly generated.*

Proof. Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be a co-truncative open quasi-compact substack, and $\mathcal{F}_U \in \text{D-mod}(U)^c$. By Proposition 1.4.6, the object $j_!(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})$ (which is well-defined by the co-truncativeness assumption) is compact. It suffices to show that such objects generate $\text{D-mod}(\mathcal{Y})$. I.e., we have to show that if $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is right-orthogonal to all such objects, then $\mathcal{F} = 0$.

For a given U, the fact that $\mathcal{F}$ is right-orthogonal to all $j_!(\mathcal{F}_U)$ as above is equivalent, by adjunction, to the fact that $j^*(\mathcal{F})$ is right-orthogonal to $\text{D-mod}(U)^c$. Since $\text{D-mod}(U)$ is compactly generated, this implies that $j^*(\mathcal{F}) = 0$. By Corollary 4.1.8, this implies that $\mathcal{F} = 0$. □

4.1.11. As was mentioned in the introduction, we will use Proposition 4.1.10 to deduce the main result of this paper, namely Theorem 0.1.2 that asserts the compact generation of $\text{D-mod}(\mathrm{Bun}_G)$ from the following:

Theorem 4.1.12. *The stack* Bun_G *is truncatable.*

This theorem will be proved in Sect. 6. Its proof uses the Harder-Narasimhan-Shatz stratification of Bun_G, but its main idea is the same as in the case when $G = SL_2$, which is considered separately in Sect. 4.4.

4.2. **Presentation as a colimit and duality.** For the rest of this section we fix $\mathcal{Y}$ to be a locally QCA truncatable algebraic stack.

4.2.1. Let us recall the following general assertion about DG categories (see [GL:DG], Lemmas 1.3.3 and 2.2.2). Let

$$i \mapsto \mathbf{C}_i,\ (i \to j) \mapsto (\phi_{i,j} \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_i, \mathbf{C}_j))$$

be a diagram of DG categories, parameterized by some index category I. Denote the corresponding functor $I \to \mathrm{DGCat}_{\mathrm{cont}}$ by Φ.

Assume that for every arrow $i \to j$, the above functor $\phi_{i,j}$ admits a left adjoint, $\psi_{j,i}$. We can then view the assignment

$$i \mapsto \mathbf{C}_i,\ (i \to j) \mapsto (\psi_{i,j} \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_j, \mathbf{C}_i)$$

as a functor $\Psi : I^{op} \to \mathrm{DGCat}_{\mathrm{cont}}$. We have:

Lemma 4.2.2.
(a) *The limit* $\mathbf{C} := \underset{\underset{I}{\longleftarrow}}{lim}\, \Phi \in \mathrm{DGCat}_{\mathrm{cont}}$ *is canonically isomorphic to the colimit*

$$\underset{\underset{I^{op}}{\longrightarrow}}{colim}\, \Psi \in \mathrm{DGCat}_{\mathrm{cont}}\,.$$

(b) *If the categories* $\mathbf{C}_i$ *are all compactly generated, then so is* $\mathbf{C}$*, and the set of its compact generators is provided by the essential images of* $\mathbf{C}_i^c$ *under the canonical functors*

$$\mathbf{C}_i \to \underset{\underset{I^{op}}{\longrightarrow}}{colim}\, \Psi \simeq \mathbf{C}.$$

Remark 4.2.3. As was mentioned earlier, limits in $\mathrm{DGCat}_{\mathrm{cont}}$ are the same as limits in DGCat. However, colimits are different. So, in Lemma 4.2.2(a) it is crucial that we take the colimit in $\mathrm{DGCat}_{\mathrm{cont}}$. For example, the colimit taken in DGCat does not have to be cocomplete.

Assume now that the categories $\mathbf{C}_i$ are dualizable. We can now produce yet another functor

$$\Phi^\vee : I^{op} \to \mathrm{DGCat}_{\mathrm{cont}}$$

that sends

$$i \mapsto \mathbf{C}_i^\vee \text{ and } (i \to j) \mapsto (\phi_{i,j}^\vee \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_j^\vee, \mathbf{C}_i^\vee)).$$

In this case we have:

Lemma 4.2.4. *The category* $\underset{\underset{I}{\longleftarrow}}{lim}\, \Phi$ *is dualizable, and its dual is given by* $\underset{\underset{I^{op}}{\longrightarrow}}{colim}\, \Phi^\vee$.

4.2.5. *The category of D-modules as a colimit.* Let us combine the assertion of Corollary 4.1.8 with that of Lemma 4.2.2(a).

We obtain:

Corollary 4.2.6. *The category* $\text{D-mod}(\mathcal{Y})$ *is canonically equivalent to*

$$\underset{\underset{U\subset \mathcal{Y}}{\longrightarrow}}{colim}\, \text{D-mod}(U),$$

where the limit is taken over the poset of co-truncative open quasi-compact substacks of $\mathcal{Y}$*, and where for* $U_\alpha \overset{j_{\alpha,\beta}}{\hookrightarrow} U_\beta$*, the functor* $\text{D-mod}(U_\alpha) \to \text{D-mod}(U_\beta)$ *is* $(j_{\alpha,\beta})_!$.

It follows from the construction that for a co-truncative open quasi-compact substack $U \overset{j}{\hookrightarrow} \mathcal{Y}$, the resulting functor $\text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$ is $j_!$.

4.2.7. *Description of the dual category.* Let us now combine Corollary 4.1.8 with Lemma 4.2.4, we obtain:

Corollary 4.2.8. *The category* $\text{D-mod}(\mathcal{Y})$ *is dualizable. Its dual category is canonically equivalent to*

$$\underset{\underset{U\subset \mathcal{Y}}{\longrightarrow}}{colim}\, \text{D-mod}(U),$$

where the colimit is taken over the poset of co-truncative open quasi-compact substacks of $\mathcal{Y}$*, and where for* $U_\alpha \overset{j_{\alpha,\beta}}{\hookrightarrow} U_\beta$*, the functor* $\text{D-mod}(U_\alpha) \to \text{D-mod}(U_\beta)$ *is* $(j_{\alpha,\beta})_*$.

We shall denote the category dual to $\text{D-mod}(\mathcal{Y})$ by $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$.

Remark 4.2.9. According to Proposition 1.4.6, compact objects in $\text{D-mod}(\mathcal{Y})$ have the property that their *-support is quasi-compact. I.e., their *-stalks vanish outside a quasi-compact subset of $\mathcal{Y}$. Similarly, compact objects of $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$ have the property that their !-stalks vanish outside a quasi-compact subset .

4.2.10. Combining Corollary 4.2.8 with Lemma 4.2.2(a), we can rewrite $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$ also as a limit:

Corollary 4.2.11. *The category* $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$ *s canonically equivalent to*

$$\underset{U\subset \mathcal{Y}}{\underset{\longleftarrow}{lim}}\ \text{D-mod}(U),$$

where the limit is taken over the poset of co-truncative open quasi-compact substacks of $\mathcal{Y}$*, and where for* $U_\alpha \overset{j_{\alpha,\beta}}{\hookrightarrow} U_\beta$*, the functor* $\text{D-mod}(U_\beta)\to \text{D-mod}(U_\alpha)$ *is* $j^?_{\alpha,\beta}$*, where the latter functor is as in Sect. 2.1.11.*

4.2.12. By construction, for every co-truncative quasi-compact open substack $U \overset{j}{\hookrightarrow} \mathcal{Y}$, we have a canonically defined functor

$$\text{D-mod}(U)\to \text{D-mod}(\mathcal{Y})_{\mathrm{co}}.$$

We denote this functor by $j_{\mathrm{co},*}$. By construction, in terms of the identifications

$$\text{D-mod}(U)^\vee \simeq \text{D-mod}(U) \text{ and } \text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y})_{\mathrm{co}},$$

the functor $j_{\mathrm{co},*}$ is the dual of $j^*:\text{D-mod}(\mathcal{Y})\to \text{D-mod}(U)$.

Similarly, from Corollary 4.2.8, we have a canonically defined functor

$$j^?:\text{D-mod}(\mathcal{Y})_{\mathrm{co}}\to \text{D-mod}(U),$$

which is the dual of $j_!:\text{D-mod}(U)\to \text{D-mod}(\mathcal{Y})$, and the right adjoint of $j_{\mathrm{co},*}$.

Lemma 4.2.13. *The functor* $j_{\mathrm{co},*}$ *is fully faithful.*

Proof. We need to show that the unit of the adjunction $\mathrm{Id}_{\text{D-mod}(U)}\to j^?\circ j_{\mathrm{co},*}$ is an isomorphism. But this is obtained by passing to dual functors in the isomorphism

$$\mathrm{Id}_{\text{D-mod}(U)}\to j^*\circ j_!.$$

□

4.2.14. By Lemma 4.2.2(b), the category $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$ is compactly generated. Its compact generators can be taken of the form $j_{\mathrm{co},*}(\mathcal{F}_U)$ for $\mathcal{F}_U\in \text{D-mod}(U)^c$.

Remark 4.2.15. Since the functors $j_{\mathrm{co},*}$ are fully faithful, it is easy to see that every compact object of $\text{D-mod}(\mathcal{Y})_{\mathrm{co}}$ is of the form $j_{\mathrm{co},*}(\mathcal{F}_U)$ for some U and $\mathcal{F}_U\in \text{D-mod}(U)^c$ as above.

4.3. **Relation between the category and its dual.**

4.3.1. By construction, the DG category $\mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(\mathcal{Y})_{\mathrm{co}}, \text{D-mod}(\mathcal{Y})$ identifies canonically with

$$(\text{D-mod}(\mathcal{Y})_{\mathrm{co}})^\vee\otimes \text{D-mod}(\mathcal{Y})\simeq \text{D-mod}(\mathcal{Y})\otimes \text{D-mod}(\mathcal{Y})\simeq \text{D-mod}(\mathcal{Y}\times\mathcal{Y}).$$

Thus, every object $\mathcal{Q}\in \text{D-mod}(\mathcal{Y}\times\mathcal{Y})$ defines a functor

$$F_{\mathcal{Q}}:\text{D-mod}(\mathcal{Y})_{\mathrm{co}}\to \text{D-mod}(\mathcal{Y}).$$

4.3.2. *The naive functor.* Note that if $\mathcal{Y}$ was quasi-compact, we have a tautological equivalence

$$\text{D-mod}(\mathcal{Y})_{\text{co}} \simeq \text{D-mod}(\mathcal{Y}).$$

Recall from Sect. 1.3.9 that the corresponding object in $\text{D-mod}(\mathcal{Y} \times \mathcal{Y})$ is $(\Delta_{\mathcal{Y}})_{\text{dR},*}(\omega_{\mathcal{Y}})$.

For a non quasi-compact truncatable $\mathcal{Y}$ we define the functor

$$\text{Id}^{\text{naive}}_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(\mathcal{Y})$$

to be given by the object

$$(\Delta_{\mathcal{Y}})_{\text{dR},*}(\omega_{\mathcal{Y}}) \in \text{D-mod}(\mathcal{Y} \times \mathcal{Y}).$$

4.3.3. *An alternative description.* Here is a tautologically equivalent expression for the functor $\text{Id}^{\text{naive}}_{\mathcal{Y}}$:

By Corollary 4.2.8, to specify a continuous functor F from $\text{D-mod}(\mathcal{Y})_{\text{co}}$ to an arbitrary DG category $\mathbf{C}$, is equivalent to specifying a compatible collection of functors $F_U : \text{D-mod}(U) \to \mathbf{C}$ for co-truncative quasi-compact open substacks $U \subset \mathcal{Y}$. The compatibility condition reads that for $U_\alpha \overset{j_{\alpha,\beta}}{\hookrightarrow} U_\beta$, we must be given a (homotopy-coherent) system of isomorphism

$$F_{U_\alpha} \simeq F_{U_\beta} \circ (j_{\alpha,\beta})_*.$$

Taking $\mathbf{C} = \text{D-mod}(\mathcal{Y})$, the corresponding functors $(\text{Id}^{\text{naive}}_{\mathcal{Y}})_U$ are

$$j_* : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$$

for $U \overset{j}{\hookrightarrow} \mathcal{Y}$.

4.3.4. *Warning.* For a general truncatable stack $\mathcal{Y}$, the functor $\text{Id}^{\text{naive}}_{\mathcal{Y}}$ is *not* an equivalence. In particular, it is *not* an equivalence for $\mathcal{Y} = \text{Bun}_G$ unless G is solvable.

In fact, we have the following assertion:

Proposition 4.3.5. *If the functor* $\text{Id}^{\text{naive}}_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y})_{\text{co}} \to \text{D-mod}(\mathcal{Y})$ *is an equivalence then the closure of any quasi-compact open substack of* $\mathcal{Y}$ *is quasi-compact.*

The converse statement is also true (for tautological reasons).

The proof of Proposition 4.3.5 given below is based on the following lemma.

Lemma 4.3.6. *Let Z be a quasi-compact scheme, $\mathcal{Y}$ a QCA algebraic stack stack, and $f : Z \to \mathcal{Y}$ a morphism. Then for any holonomic D-module $\mathcal{F}$ on Z the object* $f_{\text{dR},*}(\mathcal{F}) \in \text{D-mod}(\mathcal{Y})$ *is compact.*

Proof. Let us give two proofs:

(I) This follows from Lemma 2.1.3. Indeed, the functor $f_!$, left adjoint to $f^!$ is defined on holonomic objects.

(II) $f_{\text{dR},*}(\mathcal{F})$ is holonomic and therefore coherent. Since Z is a scheme, by Theorem 1.3.6(ii), $\mathcal{F}$ is safe. By [DrGa, Lemma 9.4.2] we obtain that $f_{\text{dR},*}(\mathcal{F})$ is also safe. Thus, $f_{\text{dR},*}(\mathcal{F})$ is coherent and safe = compact.

□

Proof of Proposition 4.3.5. Suppose that $\mathrm{Id}^{\mathrm{naive}}_{\mathcal{Y}}$ is an equivalence. Since $\mathcal{Y}$ is truncatable, it is enough to show that the closure of every co-truncative open quasi-compact substack is quasi-compact.

By assumption, the functor $\mathrm{Id}^{\mathrm{naive}}_{\mathcal{Y}}$ preserves compactness. From Sect. 4.3.3, we obtain that $\mathrm{Id}^{\mathrm{naive}}_{\mathcal{Y}}$ sends a compact object $j_{\mathrm{op},*}(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})_{\mathrm{op}}$, $\mathcal{F}_U \in \text{D-mod}(U_\alpha)^c$ with $U \overset{j}{\hookrightarrow} \mathcal{Y}$ co-truncative, to $j_*(\mathcal{F}_U) \in \text{D-mod}(\mathcal{Y})$. Thus, we obtain that $j_*(\mathcal{F}_U)$ needs to be compact for any $\mathcal{F}_U \in \text{D-mod}(U)^c$ whenever U is co-truncative.

Take $\mathcal{F}_U = f_{\mathrm{dR},*}(k_Z)$, where Z is any quasi-compact scheme equipped with a morphism $f : Z \to U$ and k_Z is the "constant sheaf" on Z. By Proposition 1.4.6, there exists a quasi-compact open substack $V \subset \mathcal{Y}$ such that the $*$-stalk of $j_{\mathrm{dR},*}(\mathcal{F}_U) = (j \circ f)_{\mathrm{dR},*}(k_Z)$ over any point of $\mathcal{Y} - V$ is zero. This means that the closure of the image of $j \circ f : Z \to \mathcal{Y}$ is contained in V and therefore quasi-compact. Taking f surjective we see that the closure of U is quasi-compact. □

4.3.7. *A better functor.* Another functor

$$\mathrm{Id}'_{\mathcal{Y}} : \text{D-mod}(\mathcal{Y})_{\mathrm{co}} \to \text{D-mod}(\mathcal{Y})$$

is constructed in [GL:Kernels, Sect. 6].

Namely, in terms of (4.3.1), it is given by the object

$$(\Delta_{\mathcal{Y}})_!(\omega_{\mathcal{Y}}) \in \text{D-mod}(\mathcal{Y} \times \mathcal{Y}).$$

(The above object is well-defined because $\omega_{\mathcal{Y}}$ is holonomic.)

Remark 4.3.8. Note that if the diagonal morphism of $\mathcal{Y}$ was a closed embedding, we would have

$$(\Delta_{\mathcal{Y}})_!(\omega_{\mathcal{Y}}) \simeq (\Delta_{\mathcal{Y}})_{\mathrm{dR},*}(\omega_{\mathcal{Y}}).$$

However, obviously, for general algebraic stacks, $\Delta_{\mathcal{Y}}$ is far from being a closed embedding. Thus, $\mathrm{Id}'_{\mathcal{Y}}$ is different from $\mathrm{Id}^{\mathrm{naive}}_{\mathcal{Y}}$ even for $\mathcal{Y}$ quasi-compact.

4.3.9. Here is a basic feature of the functor $\mathrm{Id}'_{\mathcal{Y}}$:

Lemma 4.3.10. *Let $U \overset{j}{\hookrightarrow} \mathcal{Y}$ be a co-truncative quasi-compact open substack. Then there exists a canonical isomorphism of functors* $\text{D-mod}(U) \to \text{D-mod}(\mathcal{Y})$*:*

$$\mathrm{Id}'_{\mathcal{Y}} \circ j_{\mathrm{co},*} \simeq j_! \circ \mathrm{Id}'_U .$$

Proof. It is easy to see that both functors in question are given by the object

$$\begin{aligned}(j \times \mathrm{id}_U)_!(\omega_U) \in \text{D-mod}(\mathcal{Y} \times U) \simeq \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(U) \simeq \\ \simeq \text{D-mod}(\mathcal{Y}) \otimes \text{D-mod}(U)^\vee \simeq \mathrm{Funct}_{\mathrm{cont}}(\text{D-mod}(U), \text{D-mod}(\mathcal{Y})).\end{aligned}$$

□

The meaning of this lemma is that the functor $\mathrm{Id}'_{\mathcal{Y}}$ sends objects that are *-extensions from a co-truncative open substack in $\text{D-mod}(\mathcal{Y})_{\mathrm{op}}$ to objects in $\text{D-mod}(\mathcal{Y})$ that are !-extensions (from the same open).

4.3.11. We shall not pursue the study of the functor $\mathrm{Id}'_{\mathcal{Y}}$ in this paper. Let us, however, remark the following:

For a general truncatable stack $\mathcal{Y}$, the functor $\mathrm{Id}'_{\mathcal{Y}}$ is not an equivalence, even when $\mathcal{Y}$ is quasi-compact. A counter-example can be found in [GL:Kernels, Sect. 5.3.5].

However, in [GL:self-duality] it is shown that it is an equivalence in the case $\mathcal{Y} = \mathrm{Bun}_G$ (for any curve X and any reductive G). Thus, we obtain that for Bun_G, the functor $\mathrm{Id}'_{\mathrm{Bun}(G)}$ defines an equivalence

$$\text{D-mod}(\mathrm{Bun}_G)^\vee = \text{D-mod}(\mathrm{Bun}_G)_{\mathrm{co}} \simeq \text{D-mod}(\mathrm{Bun}_G).$$

4.4. **The case of SL_2.** In this subsection we will give a proof of Theorem 4.1.12 in the case $G = SL_2$, which will be the prototype of the argument in general.

4.4.1. For a positive integer $\mathrm{Bun}_G^{\leq(n)} \subset \mathrm{Bun}_G$ be the open substack consisting of vector bundles that *do not admit* line subbundles of degree $\geq n$. It is easy to see that $\mathrm{Bun}_G^{\leq(n)}$ is quasi-compact and that their union is all of Bun_G.

Let g be the genus of X. We will show that for $n > \max(g-1, 1)$, the open substack $\mathrm{Bun}_G^{\leq(n)}$ is co-truncative.

Let $\mathrm{Bun}_G^{(n)}$ be the locally closed substack

$$\mathrm{Bun}_G^{(n)} := \mathrm{Bun}_G^{\leq(n)} - \mathrm{Bun}_G^{(\leq n-1)},$$

endowed, say, with the reduced structure.

By Proposition 2.4.10, it suffices to show that $\mathrm{Bun}_G^{(n)}$ is a truncative substack of $\mathrm{Bun}_G^{\leq(n)}$. We will do this by combining Propositions 2.3.4 and 2.5.2.

4.4.2. *Reducing to a contracting situation.* For an integer n, let Bun_B^n be the stack classifying short exact sequences

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{L}^{-1} \to 0, \tag{4.1}$$

where $\mathcal{M} \in \mathrm{Bun}_{SL_2}$, and $\mathcal{L}$ is a line bundle of degree $-n$.

Let $\mathfrak{p}^n$ denote the natural projection

$$\mathrm{Bun}_B^n \to \mathrm{Bun}_G\,.$$

By Riemann-Roch, the map $\mathfrak{p}^n$ is smooth for $n > \max(g-1, 1)$, and its image equals $\mathrm{Bun}_G^{\leq(n)}$.

By Proposition 2.3.4, it suffices to show that

$$\mathrm{Bun}_G^{(n)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^n \hookrightarrow \mathrm{Bun}_B^n$$

is truncative.

4.4.3. *Applying the contraction principle.* Consider the natural map

$$\mathfrak{q}^n : \mathrm{Bun}_B^n \to \mathrm{Bun}_T^n,$$

where Bun_T^n is the stack of T-bundles of degree n, i.e., Pic^n. Explicitly, $\mathfrak{q}^n$ sends a point as in (4.1) to $\mathcal{L} \in \mathrm{Pic}^n$.

Assume that $n > \max(g-1, 1)$. Then the above map $\mathfrak{q}^n$ realizes Bun_B^n as a vector bundle over Pic^n:

For an S-point of Pic^n given by a line bundle $\mathcal{L}$ on $S \times X$, the fiber of $\mathfrak{q}^n$ over this point is the scheme of extensions as in (4.1). Let

$$\iota_n : \mathrm{Pic}^n \hookrightarrow \mathrm{Bun}_B^n$$

be the zero section.

We have a canonical action of $\mathbb{A}^1$ on Bun_B^n by Baer multiplication, which satisfies the conditions of Sect. 2.5.1. Hence, by Proposition 2.5.2, $\mathrm{Pic}^n / \mathbb{G}_m$ is a truncative substack of $\mathrm{Bun}_B^n / \mathbb{G}_m$.

Note that the resulting action of $\mathbb{G}_m \subset \mathbb{A}^1$ on Bun_B^n is canonically isomorphic to the trivial one (however, this trivialization does not preserve the projection to Pic^n). Hence, the projection

$$\mathrm{Bun}_B^n \to \mathrm{Bun}_B^n / \mathbb{G}_m$$

admits a left inverse, which is automatically smooth and is compatible with the embedding ι_n. Hence, by Proposition 2.3.4, we obtain that

$$\mathrm{Pic}^n \hookrightarrow \mathrm{Bun}_B^n$$

is truncative.

Now, the assertion of the theorem follows from the fact that the open substacks

$$(\mathrm{Bun}_B^n - \mathrm{Pic}^n) \hookrightarrow \mathrm{Bun}_B^n \text{ and } \mathrm{Bun}_G^{(\leq n-1)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^n \hookrightarrow \mathrm{Bun}_B^n$$

coincide, and hence the closed substacks

$$\mathrm{Pic}^n \hookrightarrow \mathrm{Bun}_B^n \text{ and } \mathrm{Bun}_G^{(n)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_B^n \hookrightarrow \mathrm{Bun}_B^n$$

coincide as well.

5. Recollections from reduction theory

The goal of this section is to prepare for the proof of Theorem 4.1.12 by recalling a decomposition of Bun_G into quasi-compact locally closed substacks according to the degree of instability.

With future applications in mind, when defining these open substacks, we will remove the assumption that our ground field is of characteristic 0, unless we explicitly specify otherwise. Thus, we let G be a connected reductive group over an algebraically closed field k.

5.1. **Notation related to G.**

5.1.1. To simplify the discussion we will work with a fixed choice of a Borel subgroup $B \subset G$. Conjugacy classes of parabolics are then in a objection with the set of parabolics that contain B, called *the standard parabolics* From now now, by a parabolic we shall mean a standard parabolic.

We denote by Γ_G the set of vertices of the Dynkin diagram of G. Parabolics in G are in in bijection with subsets of Γ_G. For a parabolic P with Levi quotient M we let $\Gamma_M \subset \Gamma_G$ denote the corresponding subset; it identifies with the set of vertices of the Dynkin diagram of M.

5.1.2. Let Λ_G denote the coweight lattice of G and $\Lambda_G^{\mathbb{Q}} := \mathbb{Q} \underset{\mathbb{Z}}{\otimes} \Lambda_G$. Let $\Lambda_G^+ \subset \Lambda_G$ denote the monoid of dominant coweights and $\Lambda_G^{pos} \subset \Lambda_G$ the monoid generated by positive simple coroots. Let $\Lambda_G^{+,\mathbb{Q}}, \Lambda_G^{pos,\mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$ be the corresponding rational cones.

Let $\check{\alpha}_i$, $i \in \Gamma_G$, be the simple roots; we have:

$$\Lambda_G^{+,\mathbb{Q}} = \{\lambda \in \Lambda_G^{\mathbb{Q}} \mid \langle \check{\alpha}_i, \lambda \rangle \geq 0 \text{ for } i \in \Gamma_G\}.$$

5.1.3. Let P be a parabolic of G and M its Levi quotient. Let $Z_0(M)$ be the neutral connected component of the center of M, then $\Lambda_{Z_0(M)} \subset \Lambda_G$. Set $\Lambda_{G,P}^{\mathbb{Q}} := \Lambda_{Z_0(M)}^{\mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$. Explicitly,

$$\Lambda_{G,P}^{\mathbb{Q}} = \{\lambda \in \Lambda_G^{\mathbb{Q}} \mid \langle \check{\alpha}_i, \lambda \rangle = 0 \text{ for } i \in \Gamma_M\}.$$

Set $\Lambda_{G,P}^{+,\mathbb{Q}} := \Lambda_G^{+,\mathbb{Q}} \cap \Lambda_{G,P}^{\mathbb{Q}}$ and

$$\Lambda_{G,P}^{++,\mathbb{Q}} := \{\lambda \in \Lambda_G^{\mathbb{Q}} \mid \langle \check{\alpha}_i, \lambda \rangle = 0 \text{ for } i \in \Gamma_M \text{ and } \langle \check{\alpha}_i, \lambda \rangle > 0 \text{ for } i \notin \Gamma_M\}. \tag{5.1}$$

In other words, $\Lambda_{G,P}^{++,\mathbb{Q}}$ is the set of those elements of $\Lambda_{G,P}^{+,\mathbb{Q}}$ that are regular (i.,e., lie off the walls of $\Lambda_{G,P}^{+,\mathbb{Q}}$). Clearly

$$\Lambda_G^{+,\mathbb{Q}} = \underset{P}{\sqcup} \Lambda_{G,P}^{++,\mathbb{Q}}, \tag{5.2}$$

where the union is taken over the conjugacy classes of parabolics.

5.1.4. Note also that the inclusion $\Lambda_{G,P}^{\mathbb{Q}} \hookrightarrow \Lambda_G^{\mathbb{Q}}$ canonically splits as a direct summand: the corresponding projector $\mathrm{pr}_P : \Lambda_G^{\mathbb{Q}} \to \Lambda_{G,P}^{\mathbb{Q}}$ is defined so that

$$\ker(\mathrm{pr}_P) = \bigoplus_{i \in \Gamma_M} \mathbb{Q} \cdot \alpha_i.$$

We can view the map $\Lambda_G^{\mathbb{Q}} \to \Lambda_{G,P}^{\mathbb{Q}}$ also as follows: it comes from the map

$$\Lambda_G \simeq \Lambda_M \to \Lambda_{M/[M,M]},$$

while the map $Z_0(M) \to M/[M,M]$ is an isogeny and hence induces an isomorphism

$$\Lambda_{Z_0(M)}^{\mathbb{Q}} \to \Lambda_{M/[M,M]}^{\mathbb{Q}}.$$

5.1.5. We introduce the order relation on $\Lambda^{\mathbb{Q}}$ by

$$\lambda_1 \underset{G}{\leq} \lambda_2 \;\Leftrightarrow\; \lambda_2 - \lambda_1 \in \Lambda_G^{pos,\mathbb{Q}}.$$

The following fact is useful.[3]

Lemma 5.1.6. *For a parabolic P, the projection* pr_P *is order-preserving.*

This follows from the elementary Lemma A.2.3.

[3]We learned it from S. Schieder.

5.2. **The degree of a bundle.** Fix a connected smooth complete curve X. For any algebraic group H let Bun_H denote the stack of H-bundles on X.

5.2.1. One has a canonical isomorphism $\deg : \pi_0(\mathrm{Bun}_{\mathbb{G}_m}) \xrightarrow{\sim} \mathbb{Z}$. Accordingly, for any torus T one has a canonical isomorphism $\deg_T : \pi_0(\mathrm{Bun}_T) \xrightarrow{\sim} \Lambda_T$.

5.2.2. Let $\widetilde{G}$ be any connected affine algebraic group and let $\widetilde{G}_{\mathrm{tor}}$ be its maximal quotient torus. The composition

$$\pi_0(\mathrm{Bun}_{\widetilde{G}}) \to \pi_0(\mathrm{Bun}_{\widetilde{G}_{\mathrm{tor}}}) \xrightarrow{\deg_{\widetilde{G}_{\mathrm{tor}}}} \Lambda_{\widetilde{G}_{\mathrm{tor}}}$$

will be denoted by $\deg_{\widetilde{G}}$.

If $\widetilde{G} = G$ is reductive then $G_{\mathrm{tor}} = G/[G,G]$, and the map $Z_0(G) \to G_{\mathrm{tor}}$ is an isogeny, and hence $\Lambda^{\mathbb{Q}}_{G_{\mathrm{tor}}} \simeq \Lambda^{\mathbb{Q}}_{Z_0(G)}$. Therefore, one has a (locally constant) map $\deg_G : \mathrm{Bun}_G \to \Lambda^{\mathbb{Q}}_{Z_0(G)}$. Its fibers are not necessarily connected but have finitely many connected components; this follows from Remark 5.2.4 below.

5.2.3. Let now P be a parabolic subgroup of a reductive group G, and let M be the Levi quotient of P.

Then by Sects. 5.1.3 and 5.2.2, one has the (locally constant) maps $\deg_M : \mathrm{Bun}_M \to \Lambda^{\mathbb{Q}}_{G,P}$ and therefore $\deg_P : \mathrm{Bun}_P \to \Lambda^{\mathbb{Q}}_{G,P}$.

The preimage of $\lambda \in \Lambda^{\mathbb{Q}}_{G,P}$ in Bun_M (resp. Bun_P) is denoted by Bun^{λ}_M (resp. Bun^{λ}_P).

It is easy to see that Bun^{λ}_M and Bun^{λ}_P are empty unless λ belongs to a certain finitely generated subgroup $A_{G,P} \subset \Lambda^{\mathbb{Q}}_{G,P}$ such that $A_{G,P} \otimes \mathbb{Q} = \Lambda^{\mathbb{Q}}_{G,P}$; namely, $A_{G,P} = \mathrm{pr}_P(\Lambda_G)$.

5.2.4. *Remark.* Let $\widetilde{G}$ be any connected affine algebraic group and $\widetilde{G}_{\mathrm{red}}$ its maximal reductive quotient. Define $\pi_1(\widetilde{G})$ to be the quotient of $\Lambda_{\widetilde{G}_{\mathrm{red}}}$ by the subgroup generated by coroots. It is well known that there is a unique bijection $\pi_0(\mathrm{Bun}_{\widetilde{G}}) \xrightarrow{\sim} \pi_1(\widetilde{G})$ such that the diagram

$$\begin{tikzcd}
\pi_0(\mathrm{Bun}_{\widetilde{B}}) \arrow[r] \arrow[d] & \Lambda_{\widetilde{T}} \simeq \Lambda_{\widetilde{G}_{\mathrm{red}}} \arrow[d] \\
\pi_0(\mathrm{Bun}_{\widetilde{G}}) \arrow[r] & \pi_1(\widetilde{G})
\end{tikzcd}$$

commutes. Here $\widetilde{B}$ is a Borel subgroup of $\widetilde{G}$ and $\widetilde{T}$ is the maximal quotient torus of $\widetilde{B}$.

5.3. **Semistability.**

5.3.1. For the discussion of semi-stability, we shall often have to use the projection

$$\Upsilon_G : \Lambda^{\mathbb{Q}}_G \to \Lambda^{\mathbb{Q}}_{G_{ad}},$$

where G_{ad} is the quotient of G by its center.

5.3.2. Let $\mathfrak{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ be the natural morphism. Recall that a G-bundle $\mathcal{P}_G \in \mathrm{Bun}_G$ is called *semi-stable* if for every parabolic P such that $\mathcal{P}_G = \mathfrak{p}_P(\mathcal{P}_P)$ with $\mathcal{P}_P \in \mathrm{Bun}_P^\mu$ we have

$$\Upsilon_G(\mu) \underset{G_{ad}}{\leq} 0.$$

In fact, the notion of semi-stability can be tested just using reductions to the Borel:

Lemma 5.3.3. *A G-bundle $\mathcal{P}_G$ is semi-stable if and only if for every reduction $\mathcal{P}_B$ of $\mathcal{P}_G$ to the Borel B with $\mathcal{P}_B \in \mathrm{Bun}_B^\mu$, we have $\mu \underset{G_{ad}}{\leq} 0$.*

Proof. This follows from Lemma 5.1.6 and the fact that every M-bundle admits a reduction to the Borel of M. □

5.3.4. It is known that semi-stable bundles form an open substack $\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G$, whose intersection with each connected component of Bun_G is quasi-compact.

5.3.5. More generally, for $\theta \in \Lambda_G^{+,\mathbb{Q}}$ and a G-bundle $\mathcal{P}_G$, we say that $\mathcal{P}_G$ has *degree of semi-stability* $\underset{G}{\leq} \theta$ if for every parabolic P such that $\mathcal{P}_G = \mathfrak{p}_P(\mathcal{P}_P)$ with $\mathcal{P}_P \in \mathrm{Bun}_P^\mu$ we have

$$\mu \underset{G}{\leq} \theta.$$

As in Lemma 5.3.3, this condition is enough to check for $P = B$.

5.3.6. It is easy to see that G-bundles whose degree of semi-stability is $\underset{G}{\leq} \theta$ form a quasi-compact open substack of Bun_G; we shall denote it by $\mathrm{Bun}_G^{(\underset{G}{\leq}\theta)}$. It lies in the (finite) union of connected components of Bun_G corresponding to the image of θ under

$$\Lambda_G^{\mathbb{Q}} \to \Lambda_{G,G}^{\mathbb{Q}} \simeq \Lambda_{Z_0(G)}^{\mathbb{Q}}.$$

By definition:

$$\mathrm{Bun}_G^{ss} = \underset{\theta,\,\Upsilon(\theta)=0}{\cup} \mathrm{Bun}_G^{(\underset{G}{\leq}\theta)},$$

and for $\lambda \in \Lambda_{G,G}^{\mathbb{Q}} = \Lambda_{Z_0(G)}^{\mathbb{Q}}$, we have:

$$\mathrm{Bun}_G^{ss} \cap \mathrm{Bun}_G^\lambda = \mathrm{Bun}_G^{(\underset{G}{\leq}\lambda)}.$$

Furthermore,

$$\theta_1 \underset{G}{\leq} \theta_2 \Rightarrow \mathrm{Bun}_G^{(\underset{G}{\leq}\theta_1)} \subset \mathrm{Bun}_G^{(\underset{G}{\leq}\theta_2)},$$

and

$$\underset{\theta \in \Lambda_G^{+,\mathbb{Q}}}{\cup} \mathrm{Bun}_G^{(\underset{G}{\leq}\theta)} = \mathrm{Bun}_G.$$

5.3.7. For each parabolic $P \subset G$ with Levi quotient M we have the corresponding open substack $\mathrm{Bun}_M^{ss} \subset \mathrm{Bun}_M$; let Bun_P^{ss} denote the pre-image of Bun_M^{ss} in Bun_P.

For $\lambda \in \Lambda_{G,P}^{\mathbb{Q}}$ we let

$$\mathrm{Bun}_M^{\lambda,ss} := \mathrm{Bun}_M^{ss} \cap \mathrm{Bun}_M^\lambda = \mathrm{Bun}_M^{(\underset{M}{\leq}\lambda)},\ \mathrm{Bun}_P^{\lambda,ss} := \mathrm{Bun}_P^{ss} \cap \mathrm{Bun}_P^\lambda.$$

5.4. **The Harder-Narasimhan-Shatz stratification of** Bun_G**.** This stratification was defined in [HN, Sh, Sh2] in the case $G = GL(n)$. For any reductive G it was defined in [R1, R2, R3] and [Beh, Beh2].

5.4.1. Let us give the following definition:

Definition 5.4.2.

(i) *A morphism of schemes $f : X \to Y$ is an* almost-isomorphism *if f is finite, and each geometric fiber of f has a single point.*

(ii) *A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ is an almost-isomorphism if it is schematic, and it becomes an almost-isomorphism of schemes after any flat base change $Y \to \mathcal{Y}$ with Y being a scheme.*

5.4.3. The following is a basic result of reduction theory:

Theorem 5.4.4.

(1) *Let $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ and let $P \subset G$ be the unique parabolic such that λ belongs to the set $\Lambda_{G,P}^{++,\mathbb{Q}}$. Then $\mathfrak{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ induces an an almost-isomorphism between $\mathrm{Bun}_P^{\lambda,ss}$ and a quasi-compact locally closed reduced substack $\mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G$.*

(1') *If k has characteristic 0, then $\mathrm{Bun}_P^{\lambda,ss} \to \mathrm{Bun}_G^{(\lambda)}$ is an isomorphism.*

(2) *The substacks $\mathrm{Bun}_G^{(\lambda)}$, $\lambda \in \Lambda_G^{+,\mathbb{Q}}$, are pairwise non-intersecting, and every geometric point of Bun_G belongs to exactly one $\mathrm{Bun}_G^{(\lambda)}$.*

(3) *Let $P' \subset G$ be a parabolic and let λ' be any (not necessarily dominant) element of $\Lambda_{G,P'}^{\mathbb{Q}}$. If $\mathfrak{p}_{P'}(\mathrm{Bun}_{P'}^{\lambda'}) \cap \mathrm{Bun}_G^{(\lambda)} \neq \emptyset$ then $\lambda' \underset{G}{\leq} \lambda$.*

5.4.5. It follows from the definition of the locally closed subsets $\mathrm{Bun}_G^{(\lambda)}$ that for $\theta \in \Lambda_G^{+,\mathbb{Q}}$, we have

$$\mathrm{Bun}_G^{(\underset{G}{\leq}\theta)} = \bigcup_{\lambda,\, \lambda \underset{G}{\leq} \theta} \mathrm{Bun}_G^{(\lambda)}. \tag{5.3}$$

Furthermore, we have:

Lemma 5.4.6. *Let $S \subset \Lambda_G^{+,\mathbb{Q}}$ be a subset. Consider the corresponding subset*

$$\mathrm{Bun}_G^{(S)} := \underset{\lambda \in S}{\cup} \mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G .$$

(a) *If S has the property that $\lambda_1 \in S$, $\lambda_1 \underset{G}{\leq} \lambda \Rightarrow \lambda \in S$, then $\mathrm{Bun}_G^{(S)}$ is closed in Bun_G.*

(b) *If S has the property that $\lambda_1 \in S$, $\lambda \underset{G}{\leq} \lambda_1 \Rightarrow \lambda \in S$, then $\mathrm{Bun}_G^{(S)}$ is open in Bun_G.*

(c) *If S has the property that $\lambda_1, \lambda_2 \in S$, $\lambda_1 \underset{G}{\leq} \lambda \underset{G}{\leq} \lambda_2 \Rightarrow \lambda \in S$, then $\mathrm{Bun}_G^{(S)}$ is locally closed in Bun_G.*

In cases (a) and (c) of the lemma, we shall regard $\mathrm{Bun}_G^{(S)}$ as a substack of Bun_G with the reduced structure.

Remark 5.4.7. By Sect. 5.2.3, the set $\{\lambda \in \Lambda_G^{+,\mathbb{Q}} \mid \mathrm{Bun}_G^{(\lambda)} \neq \emptyset\}$ is discrete in $\Lambda_G^{+,\mathbb{R}} := \Lambda_G^{+} \otimes \mathbb{R}$.

5.4.8. Let us make some remarks regarding the proof of Theorem 5.4.4. A full proof along these lines will appear in [Sch].

(i) For a G-bundle $\mathcal{P}_G$ we let λ be *a* maximal element in $\Lambda_G^{\mathbb{Q}}$, with respect to the $\underset{G}{\leq}$ order relation, such that there exists a parabolic P and $\mathcal{P}_P \in \mathrm{Bun}_P^\lambda$ such that $\mathcal{P}_G = \mathfrak{p}_P(\mathcal{P}_P)$. One uses the map $\mathfrak{L}_G$ (see Sect. 6.1) to show that $\lambda \in \Lambda_G^{+,\mathbb{Q}}$, and Lemma 5.1.6 to show that $\mathcal{P}_P \in \mathrm{Bun}_P^{\lambda,ss}$.

(ii) Using Bruhat decomposition, one shows that if P' is another parabolic and $\mathcal{P}_{P'} \in \mathrm{Bun}_{P'}^{\lambda'}$ such that $\mathcal{P}_G = \mathfrak{p}_{P'}(\mathcal{P}_{P'})$, then $\lambda' \underset{G}{\leq} \lambda$, and the equality takes place if and only if $P' \subset P$ and $\mathcal{P}_P$ is induced from $\mathcal{P}_{P'}$ via the above inclusion.

(iii) We obtain that the set of *maximal elements* λ as in (i) contains a single element. Moreover, the set of parabolics as in (i) also contains a unique maximal element P; namely, one for which $\lambda \in \Lambda_{G,P}^{++,\mathbb{Q}}$.

(iv) This establishes points (2) and (3) of the theorem, modulo the fact that $\mathrm{Bun}_G^{(\lambda)}$ is locally closed, and not just constructible.

(v) Let λ and P be as in (iii). To prove point (1), we use the relative compactification

$$\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_P \to \mathrm{Bun}_G$$

of the map $\mathrm{Bun}_P \to \mathrm{Bun}_G$, defined as in [BG]. Since $\overline{\mathfrak{p}}$ is proper, the images of $\overline{\mathrm{Bun}}_P{}^\lambda$ and $\overline{\mathrm{Bun}}_P{}^\lambda - \mathrm{Bun}_P^{\lambda,ss}$ in Bun_G are both closed. Using (ii), one shows that the latter does not intersect $\mathrm{Bun}_G^{(\lambda)}$. This implies that $\mathfrak{p}$ defines a finite map from $\mathrm{Bun}_P^{\lambda,ss}$ to a locally closed substack of Bun_G. It is bijective at the level of k-points by (ii).

(vi) To prove point (1') we use the fact that in charactерristic 0, a homomorphism of reductive groups $G_1 \to G_2$ sends $\mathrm{Bun}_{G_1}^{ss}$ to $\mathrm{Bun}_{G_2}^{ss}$. This allows to show that the map $\mathfrak{p} : \mathrm{Bun}_P^{\lambda,ss} \to \mathrm{Bun}_G$ is injective on S-points for any S. Combined with (1), this yields the result.

5.4.9. For the proof of our main theorem, we shall need the following generalization of point (1) of Theorem 5.4.4.

Let P be a parabolic, and let μ be an element of $\Lambda_{G,P}^{+,\mathbb{Q}}$. Let S be a subset of

$$\mathrm{pr}_P^{-1}(\mu) \cap \Lambda_G^{+,\mathbb{Q}},$$

with the property that

$$\lambda, \lambda_1 \in \Lambda_M^{+,\mathbb{Q}},\ \lambda_1 \in S,\ \lambda \underset{M}{\leq} \lambda_1 \ \Rightarrow\ \lambda \in S.$$

(For example, all of $\mathrm{pr}_P^{-1}(\mu) \cap \Lambda_G^{+,\mathbb{Q}}$ has this property.)

Note that according to Lemma 5.4.6, the subset

$$\mathrm{Bun}_M^{(S)} := \underset{\lambda \in S}{\cup} \mathrm{Bun}_M^{(\lambda)} \subset \mathrm{Bun}_M$$

is open, and the subset

$$\mathrm{Bun}_G^{(S)} := \underset{\lambda \in S}{\cup} \mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G$$

is locally closed (and thus we can regard it as a substack in the reduced structure).

Under the above circumstances we have:

Proposition 5.4.10. *Suppose that S is such that*

$$\forall\, \lambda \in S,\ \langle \check{\alpha}_j, \lambda\rangle > 0 \text{ for } j \notin \Gamma_M.$$

Then the restriction of $\mathfrak{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ *to* $\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(S)} \subset \mathrm{Bun}_P$ *defines an almost-isomorphism*

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(S)} \to \mathrm{Bun}_G^{(S)}.$$

Proof. The fact that the map in question is bijective at the level of k-points follows from Theorem 5.4.4. The fact that it is finite follows by the same argument as in (v) in Sect. 5.4.8 using the stack $\overline{\mathrm{Bun}}_P$. □

6. Proof of the main theorem

In the case of SL_2 was saw that all but finitely many of the locally closed subsets $\mathrm{Bun}_G^{(n)}$ were truncative. This allowed to show that the open substacks $\mathrm{Bun}_G^{(\leq n)}$ are co-truncative for all but finitely many values of n.

For groups of higher rank it will no longer be true that the locally closed subsets $\mathrm{Bun}_G^{(\lambda)}$ are truncative for all but finitely many λ's. It is true, however, for λ lying "deep inside" the interior of a wall in $\Lambda^{+,\mathbb{Q}}$, but the latter fact is not sufficient for the construction of open co-truncative subsets.

What we will do is combine some of the strata $\mathrm{Bun}_G^{(\lambda)}$ into unions that are truncative. We shall show that this implies that the open subsets $\mathrm{Bun}_G^{(\underset{G}{\leq}\lambda)}$ are co-truncative as long as λ is "deep enough" inside the dominant chamber $\Lambda_G^{+,\mathbb{Q}}$. This will prove Theorem 4.1.12.

Apart from the actual proof of truncativeness, we will be working over an arbitrary algebraically closed ground field k (i.e., we do not need that k be of characteristic 0).

6.1. **The Langlands retraction.**

6.1.1. We will need the following map

$$\mathfrak{L}_G : \Lambda_G^{\mathbb{Q}} \to \Lambda_G^{+,\mathbb{Q}},$$

which was first used by R. Langlands for the classification of representations of real reductive groups in terms of tempered ones, see [La2] or [BoWa] [4].

The map $\mathfrak{L}_G$ can be uniquely characterized by the property that for $\lambda \in \Lambda_G^{\mathbb{Q}}$,

$$\lambda = \mathfrak{L}_G(\lambda) - \sum_{i\in\Gamma_G} c_i \cdot \alpha_i,$$

so that $c_i \geq 0$, and for those i for which $c_i > 0$, we have $\langle \check{\alpha}_i, \mathfrak{L}_G(\lambda)\rangle = 0$. For the proof of the existence and uniqueness of the map $\mathfrak{L}_G$ and the proofs of the properties listed below, see Appendix A (in particular, Corollary A.6.1) or references therein.

Lemma 6.1.2. *The map $\mathfrak{L}_G$ preserves the $\underset{G}{\leq}$ order relation.*

Corollary 6.1.3. *For $\lambda \in \Lambda_G^{\mathbb{Q}}$ and $\lambda' \in \Lambda_G^{+,\mathbb{Q}}$ we have*

$$\lambda \underset{G}{\leq} \lambda' \Leftrightarrow \mathfrak{L}_G(\lambda) \underset{G}{\leq} \lambda'.$$

[4] We are grateful to R. Bezrukavnikov for pointing out the above sources to us.

6.1.4. The map $\mathfrak{L}_G$ can also be characterized as follows. Choose a W-invariant scalar product on $\Lambda_G^{\mathbb{R}}$. Then for $\lambda \in \Lambda_G^{\mathbb{R}}$, the point $\mathfrak{L}_G(\lambda)$ is that minimizing the distance from λ among the points of $\Lambda_G^{+,\mathbb{R}}$.

6.2. **The construction of truncative substacks.**

6.2.1. Let $\eta \in \Lambda_G^{+,\mathbb{Q}}$ to be a coweight. Consider the subset $(\eta + \Lambda_G^{+,\mathbb{Q}}) \subset \Lambda_G^{+,\mathbb{Q}}$.

We consider a new map

$$\mathfrak{L}_{G,\eta} : \Lambda_G^{\mathbb{Q}} \to (\eta + \Lambda_G^{+,\mathbb{Q}}),$$

defined as $\mathfrak{L}_{G,\eta}(\theta) = \mathfrak{L}_G(\theta - \eta) + \eta$. I.e., this is the η-shifted version of the Langlands retraction.

It is easy to see that

$$\mathfrak{L}_{G,\eta} \circ \mathfrak{L}_G = \mathfrak{L}_{G,\eta}. \tag{6.1}$$

6.2.2. For $\theta \in (\eta + \Lambda_G^{+,\mathbb{Q}})$ consider the subset

$$\mathrm{Bun}_G^{(\theta)_\eta} := \bigcup_{\lambda \in \Lambda_G^{+,\mathbb{Q}},\, \mathfrak{L}_{G,\eta} = \theta} \mathrm{Bun}_G^{(\lambda)}. \tag{6.2}$$

It follows from Lemma 5.4.6 that $\mathrm{Bun}_G^{(\theta)_\eta}$ is a quasi-compact locally closed subset of Bun_G. We shall view it as a substack by endowing it with the reduced structure.

It is clear that for $\theta \in (\eta + \Lambda_G^{+,\mathbb{Q}})$, we have:

$$\mathrm{Bun}_G^{(\theta)_\eta} = \mathrm{Bun}_G^{(\underset{G}{\leq}\theta)} - \bigcup_{\theta_1 \in (\eta + \Lambda_G^{+,\mathbb{Q}}),\, \theta_1 \underset{G}{\leq} \theta,\, \theta_1 \neq \theta} \mathrm{Bun}_G^{(\underset{G}{\leq}\theta_1)}. \tag{6.3}$$

6.2.3. We are going to prove:

Theorem 6.2.4. *There exists a non-negative integer* $\mathbf{e}$, *such that for* $\eta \in \mathbf{e} \cdot \rho + \Lambda_G^{+,\mathbb{Q}}$, *the locally closed substacks* $\mathrm{Bun}_G^{(\theta)_\eta} \subset \mathrm{Bun}_G$ *are truncative.*

In the theorem, $\rho \in \Lambda_G^{+,\mathbb{Q}}$ is the half-sum of the positive coroots of G.

We shall give the precise estimate on $\mathbf{e}$ in Sect. 6.4.6. In characteristic 0 (which is when we can talk about truncativeness), we can take $\mathbf{e} = (2g-2)$ for $g \geq 1$ and $\mathbf{e} = 0$ for $g = 0$.

We will in fact show that these substacks are *contractive* in the sense of Definition 3.2.2.

6.2.5. Let us show how this theorem implies Theorem 4.1.12:

Proof. (of Theorem 4.1.12)

Let η as in Theorem 6.2.4. It is enough to show that the open substacks $\mathrm{Bun}_G^{(\underset{G}{\leq}\theta)}$ for $\theta \in (\eta + \Lambda_G^{+,\mathbb{Q}})$ are co-truncative.

For that it suffices to show that whenever $\theta \underset{G}{\leq} \theta_1$, the substack $\mathrm{Bun}_G^{(\underset{G}{\leq}\theta)}$ is co-truncative inside $\mathrm{Bun}_G^{(\leq\theta_1)}$. However, this follows from Theorem 6.2.4 by Proposition 2.4.10. □

6.3. **A digression: bundles for Levi subgroups.**

6.3.1. For a parabolic P with a Levi quotient M consider the projection

$$\mathrm{pt}/P \to \mathrm{pt}/M. \tag{6.4}$$

Choose a splitting $P \hookleftarrow M$ of $P \twoheadrightarrow M$. In particular, we obtain a left inverse

$$\mathrm{pt}/P \leftarrow \mathrm{pt}/M \tag{6.5}$$

of (6.4).

In addition, the splitting defines an action of M on $U(P)$, the unipotent radical of P. Hence, we obtain the stack pt/M carries a canonical relative group-scheme, denoted $U(P)_{\mathcal{P}_M^{\mathrm{univ}}}$, obtained by twisting the unipotent radical $U(P)$ by the universal M-bundle $\mathcal{P}_M^{\mathrm{univ}}$ over pt/M. We have a canonical identification:

$$\mathrm{pt}/P \simeq \mathrm{Tors}_{\mathrm{pt}/M}(U(P)_{\mathcal{P}_M^{\mathrm{univ}}}), \tag{6.6}$$

as stacks over pt/M. The map (6.5) corresponds to taking the trivial $\mathcal{P}_M^{\mathrm{univ}}$-torsor.

6.3.2. We have learned the following observation from J. Lurie:

Lemma 6.3.3. *The relative group-scheme $U(P)_{\mathcal{P}_M^{\mathrm{univ}}}$ and the identification* (6.6) *are canonically independent of the choice of a splitting $P \leftarrow M$.*

Proof. Follows from the fact that any two splittings $P \leftleftarrows M$ are *uniquely* conjugate by an element of $U(P)$. □

As a corollary, we obtain:

Corollary 6.3.4. *The map* (6.5) *is also canonically independent of the choice of a splitting $P \hookleftarrow M$.*

This, in turn, implies:

Corollary 6.3.5. *The restriction functor* $\mathrm{Rep}(P) \to \mathrm{Rep}(M)$ *is canonically defined.*

6.3.6. A choice of a splitting $P \leftarrow M$ is equivalent to a choice of an *opposite parabolic* P^- such that the composition

$$P^- \cap P \hookrightarrow P \twoheadrightarrow M$$

is an isomorphism. (Note that P^- is not a "standard parabolic", i.e., it does not contain our chosen B.)

By Lemma 6.3.3, the opposite parabolic P^- is defined up to a unique conjugacy by $U(P)$. In particular, the stack pt/P^-, equipped with the maps

$$\mathrm{pt}/G \leftarrow \mathrm{pt}/P^- \to \mathrm{pt}/M,$$

and also a section $\mathrm{pt}/M \to \mathrm{pt}/P^-$, are well-defined.

By definition pt/M identifies with an open substack of $\mathrm{pt}/P^- \underset{\mathrm{pt}/G}{\times} \mathrm{pt}/P$ corresponding to G-bundles, equipped with a pair of reductions to P and P^- that are transversal. The composed maps

$$\mathrm{pt}/M \to \mathrm{pt}/P^- \underset{\mathrm{pt}/G}{\times} \mathrm{pt}/P \to \mathrm{pt}/P \text{ and } \mathrm{pt}/M \to \mathrm{pt}/P^- \underset{\mathrm{pt}/G}{\times} \mathrm{pt}/P \to \mathrm{pt}/P^-$$

identify with the above canonical splittings

$$\mathrm{pt}/M \to \mathrm{pt}/P \text{ and } \mathrm{pt}/M \to \mathrm{pt}/P^-,$$

respectively.

6.3.7. For a curve X we consider the stacks of maps

$$\mathbf{Maps}(X, \mathrm{pt}/P) \simeq \mathrm{Bun}_P, \ \mathbf{Maps}(X, \mathrm{pt}/P^-) \simeq \mathrm{Bun}_{P^-} \text{ and } \mathbf{Maps}(X, \mathrm{pt}/M) \simeq \mathrm{Bun}_M,$$

the sections

$$\iota_P : \mathrm{Bun}_M \hookrightarrow \mathrm{Bun}_P \text{ and } \iota_{P^-} : Bun_M \hookrightarrow \mathrm{Bun}_{P^-}$$

and an open embedding

$$\mathrm{Bun}_M \hookrightarrow \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P,$$

such that the compositions

$$\mathrm{Bun}_M \hookrightarrow \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P \to \mathrm{Bun}_P \text{ and } \mathrm{Bun}_M \hookrightarrow \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P \to \mathrm{Bun}_{P^-}$$

identify with the maps ι_P and ι_{P^-}, respectively.

6.3.8. For the sake of completeness, let us make the following observation related to Theorem 5.4.4. Let λ be an element of $\Lambda_P^{++,\mathbb{Q}}$. Consider the corresponding open substack

$$\mathrm{Bun}_M^{\lambda,ss} \subset \mathrm{Bun}_M .$$

Lemma 6.3.9. *The open embedding*

$$\mathrm{Bun}_M^{\lambda,ss} \hookrightarrow \mathrm{Bun}_{P^-}^{\lambda,ss} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P^{\lambda,ss}$$

is an isomorphism.

6.4. **The estimate.** In this subsection we will specify for which values of η Theorem 6.2.4 will hold. Essentially, this boils down to smoothness of a certain map.

6.4.1. First, let us give an alternative description of the strata $\mathrm{Bun}_G^{(\theta)_\eta}$, valid for any $\eta \in \Lambda_G^{\mathbb{Q},+}$. For a given $\theta \in (\eta + \Lambda_G^{+,\mathbb{Q}})$, let P be the unique parabolic such that

$$\theta \in (\eta + \Lambda_{G,P}^{++,\mathbb{Q}}).$$

Note that we can view θ also as an element of $\Lambda_M^{\mathbb{Q},+}$. Consider the open substack

$$\mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \subset \mathrm{Bun}_M .$$

Lemma 6.4.2. *The map $\mathfrak{p}_P : \mathrm{Bun}_P \to \mathrm{Bun}_G$ gives rise to a map*

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \to \mathrm{Bun}_G^{(\theta)_\eta},$$

which is an almost-isomorphism. Under this map for $\lambda \in \Lambda_M^{+,\mathbb{Q}}$ such that

$$\mathrm{Bun}_M^{(\lambda)} \subset \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)},$$

we have $\lambda \in \Lambda_G^{+,\mathbb{Q}}$, and the image of

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\lambda)} \to \mathrm{Bun}_G$$

equals $\mathrm{Bun}_G^{(\lambda)}$.

Proof. Follows immediately from Proposition 5.4.10. $\square$

Remark 6.4.3. We will show in Corollary 6.4.10 that the map

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \to \mathrm{Bun}_G^{(\theta)_\eta}$$

is in fact an isomorphism when η is deep enough in the dominant chamber.

6.4.4. We are now ready to give an estimate on $\eta \in \Lambda_G^{+,\mathbb{Q}}$ for which Theorem 6.2.4 will hold.

Let $\widetilde{G}$ be a reductive group and V its finite-dimensional representation, on which $Z_0(\widetilde{G})$ acts by a character $\check{\mu}$.

Lemma 6.4.5. *There exists a non-negative integer* $\mathbf{e}$ *such that for every* $\mathcal{P}_{\widetilde{G}} \in \mathrm{Bun}_{\widetilde{G}}^{ss}$ *such that*

$$\begin{cases} \langle \check{\mu}, \deg_{\widetilde{G}}(\mathcal{P}_{\widetilde{G}})\rangle > \mathbf{e} \;\Rightarrow\; H^1(X, V_{\mathcal{P}_{\widetilde{G}}}) = 0, \\ \langle \check{\mu}, \deg_{\widetilde{G}}(\mathcal{P}_{\widetilde{G}})\rangle < -\mathbf{e} \;\Rightarrow\; H^0(X, V_{\mathcal{P}_{\widetilde{G}}}) = 0, \end{cases}$$

where $V_{\mathcal{P}_{\widetilde{G}}}$ *denotes the vector bundle on* X *associated with* $\mathcal{P}_{\widetilde{G}}$ *and the* $\widetilde{G}$*-module* V*.*

Proof. Follows from the fact that the intersection of $\mathrm{Bun}_{\widetilde{G}}^{ss}$ with every connected component of $\mathrm{Bun}_{\widetilde{G}}$ is quasi-compact, and that under the action of $\mathrm{Bun}_{Z_0(\widetilde{G})}$ on $\mathrm{Bun}_{\widetilde{G}}$, the number of orbits of $\pi_0(\mathrm{Bun}_{Z_0(\widetilde{G})})$ on $\pi_0(\mathrm{Bun}_{\widetilde{G}})$ is finite. □

For example, if k is of characteristic 0, we can take $\mathbf{e} = 2g-2$ if $g \geq 1$ and $\mathbf{e} = 0$ if $g = 0$; in particular, it does not depend on V. In positive characteristic it is also possible to give an explicit estimate, but the formula is more complicated.

6.4.6. Let $P' \subset P$ be a pair of parabolics. Consider the action of P' on $\mathfrak{n}(P)$, where $\mathfrak{n}(P)$ denote the Lie algebra of $U(P)$. It has a canonical filtration indexed by elements

$$\check{\mu}' \in (\Lambda_{Z_0(M')})^\vee,$$

where $(\Lambda_{Z_0(M)})^\vee$ is the dual lattice of $\Lambda_{Z_0(M')}$.

Let $\mathfrak{n}(P)_{\check{\mu}'}$ be the corresponding subquotient. The action of P' on $\mathfrak{n}(P)_{\check{\mu}'}$ factors through M', and the latter has central character $\check{\mu}'$.

Let $\mathbf{e}$ be as in Lemma 6.4.5 that works for all $P, P', \check{\mu}'$ as above.

6.4.7. We will use the above condition on $\mathbf{e}$ as follows.

Proposition 6.4.8. *Let* η *be an element of* $\mathbf{e}\cdot\rho + \Lambda_G^{+,\mathbb{Q}}$*, and let* θ *and* P *be as in Sect. 6.4.1.*

(a) *The map* $\iota_P : \mathrm{Bun}_M \to \mathrm{Bun}_P$ *is a* unipotent gerbe [5] *over* $\mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \subset \mathrm{Bun}_M$*. In particular, this map is smooth and surjective.*

(b) *The map* $\mathfrak{p}_{P^-} : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G$ *is smooth when restricted to*

$$\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \subset \mathrm{Bun}_{P^-}.$$

(c) *The map* $\mathfrak{q}_{P^-} : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M$ *is schematic and affine over*

$$\mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \subset \mathrm{Bun}_M.$$

[5] See [DrGa], Definition 9.3.4 for the notion of unipotent gerbe.

Proof. For an S-point $\mathcal{P}_M$ of Bun_M, consider the corresponding group-scheme $U(P)_{\mathcal{P}_M}$ on $S \times X$. The fiber of the map $\mathfrak{q}_P$ over $\mathcal{P}_M$ is the stack of $U(P)_{\mathcal{P}_M}$-torsors.

Since $U(P)_{\mathcal{P}_M}$ is unipotent, in order to prove point (a), it suffices to show that if $\mathcal{P}_M$ hits $\mathrm{Bun}_M^{(\leq \theta)}$, then $H^1(S \times X, \mathfrak{n}(P)_{\mathcal{P}_M}) = 0$. This condition is enough to check when $S = \mathrm{Spec}(k)$. By the long exact sequence, it is sufficient to check that

$$H^1(X, (\mathfrak{n}(P)_{\check{\mu}})_{\mathcal{P}_M}) = 0$$

for every $\check{\mu} \in (\Lambda_{Z_0(M')})^\vee$.

Similarly, to prove point (b), we need to show that for every k-point $\mathcal{P}_{P^-}$ of Bun_{P^-} such that $\mathfrak{q}_{P^-}(\mathcal{P}_{P^-}) =: \mathcal{P}_M \in \mathrm{Bun}_M^{(\leq \theta)}$, we have

$$H^1(X, (\mathfrak{g}/\mathfrak{p}^-)_{\mathcal{P}_P}) = 0.$$

Again, by the long exact sequence, we obtain that it is sufficient to check that

$$H^1(X, (\mathfrak{n}(P)_{\check{\mu}})_{\mathcal{P}_M}) = 0$$

for every $\check{\mu} \in (\Lambda_{Z_0(M')})^\vee$.

By the same logic, to prove (c), it is enough to show that for $\mathcal{P}_M$ as above,

$$H^0(X, (\mathfrak{n}(P^-)_{\check{\mu}})_{\mathcal{P}_M}) = 0.$$

Let us show that under the above vanishing of H^0 and H^1 holds under our conditions on θ.

Let $\lambda \in \Lambda_M^{+,\mathbb{Q}}$ be such that $\mathcal{P}_M \in \mathrm{Bun}_M^{(\lambda)}$. Let P'_M be the corresponding parabolic in M, and let $P' \subset P$ be its preimage in P. By definition, λ belongs to

$$\Lambda^{\mathbb{Q},++}_{M,P'_M} \subset \Lambda^{\mathbb{Q}}_{M,P'_M} \simeq \Lambda^{\mathbb{Q}}_{G,P'},$$

the bundle $\mathcal{P}_M$ is induced from a P'_M-bundle $\mathcal{P}_{P'_M}$, and the induced M'-bundle $\mathcal{P}_{M'}$ belongs to $\mathrm{Bun}_{M'}^{\lambda,ss}$.

We need to check the condition

$$\langle \check{\mu}', \lambda \rangle > \mathbf{e}$$

for every central character $\check{\mu}'$ of M' that appears in $\mathfrak{n}(P)$.

We have $\lambda = \theta - \lambda'$, where $\lambda' = \sum\limits_{i \in \Gamma_M} a_i \cdot \alpha_i$ with $a_i \in \mathbb{Q}^{\geq 0}$. Furthermore,

$$\theta = \eta + (\theta - \eta),$$

where $\theta - \eta \in \Lambda^{++,\mathbb{Q}}_{G,P}$.

Every $\check{\mu}'$ as above is the image of a positive root $\check{\beta}$ which can be written as $\check{\mu}_1 + \check{\mu}_2$ with

$$\check{\mu}_1 = \sum_{i \in \Gamma_M} b_i \cdot \check{\alpha}_i, \quad \check{\mu}_2 = \sum_{j \notin \Gamma_M} c_j \cdot \check{\alpha}_j$$

with $b_i, c_j \in \mathbb{Q}^{\geq 0}$, and $\check{\mu}_2 \neq 0$.

Hence,

$$\langle \check{\mu}', \lambda \rangle = \langle \check{\beta}, \lambda \rangle \geq \langle \check{\mu}_2, \lambda \rangle = \langle \check{\mu}_2, \eta \rangle + \langle \check{\mu}_2, \theta - \eta \rangle - \langle \check{\mu}_2, \lambda' \rangle > \langle \check{\mu}_2, \eta \rangle \geq \mathbf{e},$$

as required.

□

6.4.9. We are now ready to prove the following sharpened version of Lemma 6.4.2:

Corollary 6.4.10. *Let η and θ be as in Proposition 6.4.8. Then the map*

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \to \mathrm{Bun}_G^{(\theta)_\eta}$$

is an isomorphism.

Proof. Let us base change the map

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \to \mathrm{Bun}_G$$

by

$$\mathfrak{p}_{P^-} : \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \to \mathrm{Bun}_G .$$

Since the latter is smooth, it suffices to show that

$$\left(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}\right) \underset{\mathrm{Bun}_G}{\times} \left(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}\right)$$

contains an open substack U such that:

- The map
$$U \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}$$
is a locally closed embedding.
- The map
$$U \to \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}$$
is surjective.

We take U to be the image of $\mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}$ under

$$\mathrm{Bun}_M \hookrightarrow \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P .$$

The two required properties follow, respectively, from points (c) and (a) of Proposition 6.4.8. $\square$

Remark 6.4.11. It follows from Lemma 6.3.9 that the open embedding

$$\mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \hookrightarrow \left(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}\right) \underset{\mathrm{Bun}_G}{\times} \left(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}\right)$$

that we used in the lemma is in fact an isomorphism.

6.5. **Proof of truncativeness.** We shall prove Theorem 6.2.4 by reducing it to the situation of Proposition 2.5.2 via Proposition 6.4.8.

6.5.1. Consider the following general situation. Let H be an algebraic group that acts by automorphisms on the identity map of some stack $\mathcal{Q}$. Let $\widetilde{\mathcal{Q}}$ be another stack equipped with a map $f:\widetilde{\mathcal{Q}}\to\mathcal{Q}$. We thus obtain an action of H on $\widetilde{\mathcal{Q}}$ by automorphisms, such that the map f is equivariant with respect to the *trivial* action of H on $\mathcal{Q}$.

I.e., the H-action on $\widetilde{\mathcal{Q}}$ as an abstract stack is canonically trivial, but this trivialization is not compatible with the map f. The discrepancy is given by the initial action of H on the identity map of $\mathcal{Q}$.

We apply this to $\mathcal{Q}=\operatorname{pt}/M$ and $H=Z_0(M)$. Thus, we obtain a $Z_0(M)$-action on pt/P such that the map $\operatorname{pt}/P\to\operatorname{pt}/M$ is equivariant.

Lemma 6.5.2. *Let $\mathbb{G}_m\to Z_0(M)$ be a one-parameter subgroup that corresponds to an element $\mu\in\Lambda^{++,\mathbb{Q}}_{G,P}$. Then the resulting action of $\mathbb{G}_m$ on pt/P over pt/M canonically extends to an action of the monoid $\mathbb{A}^1$, such that the action of $\{0\}\in\mathbb{A}^1$ acts as*

$$\operatorname{pt}/P\to\operatorname{pt}/M\to\operatorname{pt}/P,$$

where the second arrow is given by Lemma 6.3.3.

6.5.3. Taking the stacks of maps from X to one in Lemma 6.5.2, we obtain that given a one-parameter subgroup $\mathbb{G}_m\to Z_0(M)$ corresponding to $\mu\in\Lambda^{++,\mathbb{Q}}_{G,P}$, we have a canonically defined action of the monoid $\mathbb{A}^1$ on Bun_P, such that

(i) The action of $\{0\}\in\mathbb{A}^1$ equals $\operatorname{Bun}_P\xrightarrow{\mathfrak{q}_P}\operatorname{Bun}_M\xrightarrow{\iota_P}\operatorname{Bun}_P$.

(ii) The action of $\mathbb{G}_m$ on Bun_P is canonically isomorphic to the trivial action, in a way compatible with the embedding ι_P.

6.5.4. Recall the set-up of Sect. 2.5.1.

We take

$$\mathcal{X}:=\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}\subset\operatorname{Bun}_M$$

and we take

$$\mathcal{W}:=\operatorname{Bun}_{P^-}\underset{\operatorname{Bun}_M}{\times}\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}\subset\operatorname{Bun}_{P^-}$$

with the map π being $\mathfrak{q}_{P^-}$.

By Proposition 6.4.8(a) and Sect. 6.5.3, we obtain that the conditions of Proposition 2.5.2 are satisfied. Hence,

$$\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}/\mathbb{G}_m\overset{\iota_P}{\hookrightarrow}\operatorname{Bun}_{P^-}\underset{\operatorname{Bun}_M}{\times}\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}/\mathbb{G}_m$$

is truncative.

Moreover, since the action of $\mathbb{G}_m$ on the stacks in question is trivial, the map

$$\operatorname{Bun}_{P^-}\underset{\operatorname{Bun}_M}{\times}\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}\to\operatorname{Bun}_{P^-}\underset{\operatorname{Bun}_M}{\times}\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}/\mathbb{G}_m$$

admits a canonical left inverse, compatible with ι_P, and which is automatically smooth.

Hence, by Proposition 2.3.4,

$$\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}\overset{\iota_P}{\hookrightarrow}\operatorname{Bun}_{P^-}\underset{\operatorname{Bun}_M}{\times}\operatorname{Bun}_M^{(\underset{M}{\leq}\theta)}$$

is also truncative.

6.5.5. We are finally ready to prove Theorem 6.2.4. As in the proof of Corollary 6.4.10, we base change the map

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \to \mathrm{Bun}_G$$

by

$$\mathfrak{p}_{P^-} : \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \to \mathrm{Bun}_G\,.$$

By Proposition 2.3.4 it suffices to find an open substack U in

$$\left(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}\right) \underset{\mathrm{Bun}_G}{\times} \left(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}\right)$$

with the same properties as in the proof of Corollary 6.4.10, such that the locally closed embedding

$$U \hookrightarrow \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}$$

is truncative.

We take U to be the same as in Corollary 6.4.10, i.e., the image of $\mathrm{Bun}_M^{(\underset{M}{\leq}\theta)}$ under

$$\mathrm{Bun}_M \hookrightarrow \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P\,.$$

The required truncativeness property has been established in Sect. 6.5.4 above.

□

6.5.6. *An alternative argument.* Let us reprove Theorem 6.2.4 using Theorem 3.2.3 instead of Proposition 2.5.2. Namely, we will show that the locally closed substack $\mathrm{Bun}_G^{(\theta)_\eta} \subset \mathrm{Bun}_G$ is *contractive* in the sense of Definition 3.2.2.

We apply the set-up of Theorem 3.2.3 with

$$\mathcal{X} := \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)},\ \mathcal{Z} := \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\underset{M}{\leq}\theta)} \simeq \mathrm{Bun}_G^{(\theta)_\eta},\ \mathcal{Y} := \mathrm{Bun}_G,$$

and $f = \iota_P$. The $\mathbb{G}_m$ action on ι_P comes from the $\mathbb{G}_m$-action on Bun_M defined as follows:

The action of $Z_0(M)$ by automorphisms of the identity map on pt/M induces one on Bun_M. We choose a one-parameter subgroup $\mathbb{G}_m \to Z_0(M)$ corresponding to $\mu \in (\Lambda_{G,P}^{++,\mathbb{Q}})$. Thus, we obtain a $\mathbb{G}_m$-action on the identity map of Bun_M.

We need to calculate the weights of $\mathbb{G}_m$ on the pullback to $\mathcal{Z}$ of the conormal $\mathcal{N}^*_{\mathcal{Z}/\mathcal{Y}}$. The sheaf in question is a vector bundle, whose fiber at a k-point $\mathcal{P}_M$ is the vector space dual to

$$H^1(X, (\mathfrak{g}/\mathfrak{p})_{\mathcal{P}_M}).$$

The weights of $\mathbb{G}_m$ are obtained by pairing μ with roots of G that are not roots of M, and the corresponding integers are all strictly positive.

□

7. Counterexamples

The goal of this section is to show that the property of being truncatable is a purely "stacky" phenomenon, i.e., that it it typically fails for non quasi-compact schemes.

7.1. **Formulation of the theorem.**

Theorem 7.1.1. *Let Y be an irreducible smooth scheme of dimension n, such that for some (or equivalently, for any) non-empty quasi-compact open $U \subset Y$ the set*

$$\{y \in Y - U \mid \dim_y(Y-U) = \dim Y - 1\} \tag{7.1}$$

is not quasi-compact. Then $\text{D-mod}(Y)$ *is not compactly generated.*

The theorem will be proved in Sect. 7.2 below. Here are two examples of schemes Y satisfying the condition of Theorem 7.1.1.

Example 7.1.2. Let I be an infinite set and let Y be the non-separated curve that one obtains from $\mathbb{A}^1 \times I$ by gluing together the open subschemes $(\mathbb{A}^1 - \{0\}) \times \{i\}$, $i \in I$ (in other words, Y is the affine line with the point 0 repeated I times).

Example 7.1.3. Let X_0 be a smooth surface and $x_0 \in X_0$ a point. Set $U_0 = X - \{x_0\}$. Let X_1 be the blow-up of X_0 at x_0. Let $x_1 \in X_1$ be a point on the exceptional divisor. We have an open embedding

$$U_0 = X - \{x_0\} \hookrightarrow X_1 - \{x_1\} = U_1$$

such that $U_1 - U_0$ is a divisor. We can now apply the same process for (X_1, x_1) instead of (X_0, x_0). Thus we obtain a sequence of schemes

$$U_0 \hookrightarrow U_1 \hookrightarrow U_2 \hookrightarrow \dots$$

Then $Y := \bigcup\limits_i U_i$ satisfies the condition of Theorem 7.1.1. Note that Y is separated if X_0 is.

7.2. **Proof of Theorem 7.1.1.** We will use facts from Subsect. 1.3.5 about the relation between compactness and coherence (in the easier case of smooth schemes).

7.2.1. Let Y be a smooth scheme, $Z \subset Y$ a non-empty divisor, and $Y - Z = U \overset{j}{\hookrightarrow} Y$ be the complementary opem embedding.

Lemma 7.2.2. *Suppose that $\mathcal{N} \in \text{D-mod}(Y)$ is coherent and $j_* \circ j^*(\mathcal{N}) = \mathcal{N}$. Then the singular support $SS(\mathcal{N}) \subset T^*(Y)$ is* not *equal to $T^*(Y)$.*

Proof. We can assume that Y is affine and Z is smooth. Since j_* is exact we can also assume that $\mathcal{N}$ is a D-module. Suppose that $SS(\mathcal{N}) = T^*(Y)$. Then there exists a monomorphism $\mathcal{D}_Y \hookrightarrow \mathcal{N}$. It induces a monomorphism $j_* \circ j^*(\mathcal{D}_Y) \hookrightarrow j_* \circ j^*(\mathcal{N}) = \mathcal{N}$. But $\mathcal{N}$ is coherent while $j_* \circ j^*(\mathcal{D}_Y)$ is not. □

7.2.3. Let Y be as in Theorem 7.1.1 and $\mathcal{M} \in \text{D-mod}(Y)$ a compact object. Note that by Remark 1.3.8, $\mathcal{M}$ is automatically coherent.

We claim:

Lemma 7.2.4. *$SS(\mathcal{M}) \neq T^*(Y)$.*

Proof. By Proposition 1.4.6, there exists a quasi-compact open $U \overset{j}{\hookrightarrow} Y$ such that $\mathcal{M} = j_!(j^*(\mathcal{M}))$ or equivalently,

$$\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M}) = j_* \circ j^*(\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M})).$$

We can assume that $U \neq \emptyset$ (otherwise $\mathcal{M} = 0$ and $SS(\mathcal{M}) = \emptyset$). Then the set (7.1) is non-empty, so after shrinking Y we can assume that the set $Z := Y - U$ is a non-empty divisor.

Applying Lemma 7.2.2 to $\mathcal{N} = \mathbb{D}_Y^{\text{Verdier}}(\mathcal{M})$ we get $SS(\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M})) \neq T^*(Y)$. Finally, $SS(\mathcal{M}) = SS(\mathbb{D}_Y^{\text{Verdier}}(\mathcal{M}))$. □

7.2.5. Recall that the full subcategory of compact objects in a DG category $\mathbf{C}$ is denoted by $\mathbf{C}^c$.

Lemma 7.2.6. *Let $\mathcal{A} \subset \text{D-mod}(Y)$ be the DG subcategory generated by $\text{D-mod}(Y)^c$. If $\mathcal{M} \in \mathcal{A}$ is coherent then $SS(\mathcal{M}) \neq T^*(Y)$.*

Proof. Let $U \overset{j}{\hookrightarrow} Y$ be a non-empty quasi-compact open subset.

Let $\mathbf{C} \subset \text{D-mod}(U)$ be the full DG subcategory of $\text{D-mod}(U)$ generated by $j^*(\text{D-mod}(Y)^c)$. Since $j^*(\text{D-mod}(Y)^c) \subset \text{D-mod}(U)^c$, we have

$$\mathbf{C}^c = \mathbf{C} \cap \text{D-mod}(U)^c,$$

and by Sect. 0.8.8, the latter is Karoubi-generated by $j^*(\text{D-mod}(Y)^c)$.

This observation, combined with Lemma 7.2.4 and the fact that $T^*(U)$ is dense in $T^*(Y)$ implies that for any $\mathcal{N} \in \mathbf{C}^c$,

$$SS(\mathcal{N}) \neq T^*(U).$$

Now, $j^*(\mathcal{M}) \in \mathbf{C} \cap \text{D-mod}_{\text{coh}}(U)$, and since U is quasi-compact, we have $\text{D-mod}_{\text{coh}}(U) = \text{D-mod}(U)^c$. Hence, $j^*(\mathcal{M}) \in \mathbf{C}^c$, implying the assertion of the lemma.

$\square$

Corollary 7.2.7. *The DG category $\mathcal{A}$ from Lemma 7.2.6 does not contain $\mathcal{D}_Y$.*

Theorem 7.1.1 clearly follows from Corollary 7.2.7.

Appendix A. Langlands' retraction

Given a root system in a Euclidean space V, Langlands defined in [La2, Sect. 4] a certain retraction $\mathfrak{L} : V \to V^+$, where V^+ is the dominant chamber. Later this retraction was discussed in [BoWa, Ch. IV, Subsect. 3.3] and [C, Sect. 1].

In this section we briefly recall the definition and properties of $\mathfrak{L}$. Following [C, Sect. 1], we start with the most naive definition of $\mathfrak{L}$, which makes sense for a Euclidean space equipped with *any* basis $\{\alpha_i\}$. Starting with Subsection A.2, we assume that $\langle \alpha_i , \alpha_j \rangle \leq 0$ for $i \neq j$. The key point is that under this assumption $\mathfrak{L}$ can be characterized in terms of the ordering on V (see Corollary A.2.2). It is this characterization that is important for us (and probably for other applications). In Sect. A.7 we make some historical remarks.

A.1. **The retraction defined by the metric.** Let V be a finite-dimensional vector space over $\mathbb{R}$ with a positive definite scalar product $\langle\ ,\ \rangle$. Let $\{\alpha_i\}_{i\in\Gamma}$ be an arbitrary basis in V and $\{\omega_i\}_{i\in\Gamma}$ the dual basis. Let $V^+ \subset V$ denote the closed convex cone generated by the ω_i's, $i \in \Gamma$.

Following J. Carmona [C, Sect. 1], we define the *Langlands retraction* $\mathfrak{L} : V \to V^+$ as follows: $\mathfrak{L}(x)$ is the point of V^+ closest to x (such point exists and is unique because V^+ is closed and convex). It is easy to see that the map $\mathfrak{L}$ is continuous.

Let us give another description of $\mathfrak{L}$. For a subset $J \subset \Gamma$ let K_J denote the closed convex cone generated by ω_j for $j \in \Gamma - J$ and by $-\alpha_i$ for $i \in J$. Clearly, each K_J is a simplicial cone of full dimension in V. Let V_J denote the linear span of α_j, $j \in J$ (so $V_J^\perp$ is spanned by ω_i, $i \notin J$). Let $\mathrm{pr}_J : V \to V$ denote the orthogonal projection onto $V_J^\perp$, so $\ker(\mathrm{pr}_J) = V_J$.

Proposition A.1.1. (a) *The map $\mathfrak{L}$ is piecewise linear. The cones K_J are exactly the linearity domains of $\mathfrak{L}$. For $x \in K_J$ one has $\mathfrak{L}(x) = \mathrm{pr}_J(x)$.*

(b) *The cones K_J and their faces form a complete simplicial fan*[6] *in V, combinatorially equivalent to the coordinate fan*[7].

Remark A.1.2. The wording in the above proposition was suggested to us by A. Zelevinsky.

The proposition immediately follows from the next lemma, whose proof is straightforward.

Lemma A.1.3. *Let $x \in V$ and $y \in V^+$. Set $J := \{j \in \Gamma \,|\, \langle \alpha_j , y\rangle = 0\}$. Then the following are equivalent:*
(a) *$y = \mathfrak{L}(x)$.*
(b) *$x - y$ belongs to the the closed convex cone generated by $-\alpha_j$ for $j \in J$.* □

A.2. **The key statements.** Let V^{pos} denote the cone dual to V^+, i.e., the closed convex cone generated by the α_i's, $i \in \Gamma$. Equip V and V^+ with the following partial ordering: $x \leq y$ if $y - x \in V^{pos}$. By Lemma A.1.3, the retraction $\mathfrak{L} : V \to V^+$ from Sect. A.1 has the following property:

$$\mathfrak{L}(x) \geq x, \quad x \in V. \tag{A.1}$$

Theorem A.2.1. *Assume that*

$$\langle \alpha_i , \alpha_j \rangle \leq 0 \text{ for } i \neq j. \tag{A.2}$$

Then the retraction $\mathfrak{L} : V \to V^+$ is order-preserving.

By (A.1), Theorem A.2.1 implies the following statement, which characterizes $\mathfrak{L}$ in terms of the order relation.

Corollary A.2.2. *If* (A.2) *holds then $\mathfrak{L}(x)$ is the least element in $\{y \in V^+ \,|\, y \geq x\}$.* □

Let us prove Theorem A.2.1. To show that a piecewise linear map is order-preserving it suffices to check that this is true on each of its linearity domains. So Theorem A.2.1 follows from Proposition A.1.1(a) and the next lemma.

Lemma A.2.3. *Assume* (A.2). *Then for each subset $J \subset \Gamma$ the map $\mathrm{pr}_J : V \to V$ defined in Sect. A.1 is order-preserving.*

A.3. **Proof of Lemma A.2.3.**

Lemma A.3.1. *Let $J \subset \Gamma$. Suppose that $x \in V_J$ and $\langle x , \alpha_j \rangle \geq 0$ for all $j \in J$. Then $x \geq 0$.*

Proof. We can assume that $J = \Gamma$ (otherwise replace V by V_J and Γ by Γ_J). Then the lemma just says that $V^+ \subset V^{pos}$. This is a well known consequence of (A.2). □

Now let us prove Lemma A.2.3. We have to show that $\mathrm{pr}_J(\alpha_i) \geq 0$ for any $i \in \Gamma$. If $i \in J$ then $\mathrm{pr}_J(\alpha_i) = 0$. Now suppose that $i \notin J$. By the definition of pr_J, we have $\mathrm{pr}_J(\alpha_i) = \alpha_i + x$, where x is the element of V_J such that $\langle x , \alpha_j \rangle = -\langle \alpha_i , \alpha_j \rangle$ for all $j \in J$. By (A.2) and Lemma A.3.1, $x \geq 0$, so $\mathrm{pr}_J(\alpha_i) = \alpha_i + x \geq 0$. □

[6] This means that these cones cover V and each intersection $K_J \cap K_{J'}$ is a face in both K_J and $K_{J'}$.
[7] The coordinate fan is what one gets when the basis $\{\alpha_i\}$ is orthogonal.

A.4. **Another approach to the Langlands retraction.** Suppose that (A.2) holds. Then one could take Corollary A.2.2 as the *definition* of the Langlands retraction $\mathfrak{L} : V \to V^+$, i.e., one could define $\mathfrak{L}(x)$ to be the least element of the set $\{y \in V^+ \,|\, y \geq x\}$. This set is closed and non-empty (because (A.2) implies that $V^+ \subset V^{pos}$), so the existence of the least element in it follows from the next proposition.

Proposition A.4.1. *Suppose that $\langle \alpha_i \,, \alpha_j \rangle \leq 0$ for $i \neq j$. Then the infinum of any non-empty subset of V^+ belongs to V^+.*

Here "infinum" is understood in terms of the partial ordering defined by V^{pos}. In other words, given a family of vectors

$$x_t \in V, \qquad x_t = \sum_i x_{i,t} \cdot \alpha_i \,, \tag{A.3}$$

its infinum equals $\sum\limits_i y_i \cdot \alpha_i$, where $y_i := \inf\limits_t x_{i,t}$. Note that if $x_t \in V^+$ then $x_t \in V^{pos}$, so $x_{i,t} \geq 0$ and $\inf\limits_t x_{i,t}$ exists.

Proof of Proposition A.4.1. Suppose that we have a family of vectors $x_t \in V^+$ and $y = \inf\limits_t x_t$. The assumption $x_t \in V^+$ means that $\langle x_t \,, \alpha_i \rangle \geq 0$ for all i. We have to show that $\langle y \,, \alpha_i \rangle \geq 0$ for all i.

Fix i. Write $x_t = x'_t + x''_t$, $y = y' + y''$, where

$$x'_t, y' \in \mathbb{R}\alpha_i, \quad x''_t, y'' \in \bigoplus_{j \neq i} \mathbb{R}\alpha_j \,.$$

Clearly $y' = \inf\limits_t x'_t$, $y'' = \inf\limits_t x''_t$. Then for every t one has

$$\langle x'_t \,, \alpha_i \rangle = \langle x_t \,, \alpha_i \rangle - \langle x''_t \,, \alpha_i \rangle \geq -\langle x''_t \,, \alpha_i \rangle \geq -\langle y'' \,, \alpha_i \rangle$$

(the second inequality holds because $-\langle \alpha_j \,, \alpha_i \rangle \geq 0$ for $j \neq i$). So

$$\langle y' \,, \alpha_i \rangle = \inf_t \langle x'_t \,, \alpha_i \rangle \geq -\langle y'' \,, \alpha_i \rangle,$$

i.e., $\langle y \,, \alpha_i \rangle \geq 0$. □

Remark A.4.2. In the situation of the following example Proposition A.4.1 just says that the infinum of any family of concave functions is concave. In fact, the above proof of Proposition A.4.1 is identical to the proof of this classical statement.

Example A.4.3. Consider the root system of $SL(n)$. In this case V is the orthogonal complement of the vector $\varepsilon_1 + \ldots + \varepsilon_n$ in the Euclidean space with orthonormal basis $\varepsilon_1, \ldots \varepsilon_n$, and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq n-1$. Let $\omega_i \in V$ be the basis dual to α_i. For each $v \in V$ define $f_v : \{0, \ldots, n\} \to \mathbb{R}$ by

$$f_v(0) = f_v(n) = 0, \quad f_v(i) = \langle v \,, \omega_i \rangle \quad \text{ for } 0 < i < n.$$

Then the map $v \mapsto f_v$ identifies V with the space of functions $f : \{0, \ldots, n\} \to \mathbb{R}$ such that $f_v(0) = f_v(n) = 0$. Moreover, V^{pos} identifies with the space of non-negative functions and V^+ with the space of *concave* functions. Thus the Langlands retraction assigns to a function $f : \{0, \ldots, n\} \to \mathbb{R}$ the smallest concave function which is $\geq f$.

A.5. **Rationality.** Suppose that $\langle \alpha_i , \alpha_j \rangle \in \mathbb{Q}$ for all $i, j \in \Gamma$. Then the $\mathbb{Q}$-linear span of the $\alpha_i' s$ equals the $\mathbb{Q}$-linear span of the $\omega_i' s$. Denote it by $V^{\mathbb{Q}}$. Then $V = V^{\mathbb{Q}} \otimes \mathbb{R}$. The cones K_J, the subspaces V_J, and the operators pr_J from Sect. A.1 are clearly defined over $\mathbb{Q}$. So by Proposition A.1.1,

$$\mathfrak{L}(V^{\mathbb{Q}}) \subset V^{\mathbb{Q}}. \tag{A.4}$$

A.6. **Reductive groups.** Let G be a (connected) reductive group. Let Λ_G be its coweight lattice, i.e., $\Lambda_G = \mathrm{Hom}(\mathbb{G}_m, T)$, where T is *the* maximal torus of G. Set $\Lambda_G^{\mathbb{Q}} := \Lambda_G \otimes \mathbb{Q}$. We have the simple coroots $\alpha_i \in \Lambda_G$ and the simple roots $\check{\alpha}_i \in \mathrm{Hom}(\Lambda_G, \mathbb{Z})$. Let $\Lambda_G^{+,\mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$ denote the dominant cone. Equip $\Lambda_G^{+,\mathbb{Q}}$ with the following partial ordering: $\lambda_1 \underset{G}{\leq} \lambda_2$ if $\lambda_2 - \lambda_1$ is a linear combination of the simple coroots with non-negative coefficients.

Now define the *Langlands retraction* $\mathfrak{L}_G : \Lambda_G^{\mathbb{Q}} \to \Lambda_G^{+,\mathbb{Q}}$ as follows: $\mathfrak{L}_G(\lambda)$ is the smallest element of the set

$$\{\mu \in \Lambda_G^{+,\mathbb{Q}} \mid \mu \underset{G}{\geq} \lambda\}. \tag{A.5}$$

Corollary A.6.1. *(i) $\mathfrak{L}_G(\lambda)$ exists.*

(ii) $\mathfrak{L}_G(\lambda)$ is the element of $\Lambda_G^{+,\mathbb{Q}}$ closest to λ with respect to any positive scalar product on $\Lambda_G^{+,\mathbb{Q}} \otimes \mathbb{R}$ which is invariant with respect to the Weyl group.

(iii) $\mathfrak{L}_G(\lambda)$ is the unique element of the set (A.5) *with the following property: $\langle \mathfrak{L}_G(\lambda) , \check{\alpha}_i \rangle = 0$ for any simple root $\check{\alpha}_i$ such that the coefficient of α_i in $\mathfrak{L}_G(\lambda) - \lambda$ is nonzero.*

Proof. Combine Lemma A.1.3, Corollary A.2.2, and the inclusion (A.4). □

A.7. **Some historical remarks.** In [La2] R. Langlands defined the retraction $\mathfrak{L}$ and formulated his "geometric lemmas" (see [La2, Lemmas 4.4-4.5 and Corollary 4.6]) for the purpose of the classification of representations of real reductive groups in terms of tempered ones. However, much earlier he had formulated a closely related (and more complicated) combinatorial lemma[8] in his theory of Eisenstein series, see [La1, Sect. 8]. In this work Langlands considers Eisenstein series on quotients of the form $G(\mathbb{R})/\Gamma$, where G is a reductive group over $\mathbb{Q}$ and Γ is an arithmetic subgroup, but the same technique applies to quotients of the form $G(\mathbb{A})/G(\mathbb{Q})$. So the fact that we are using the Langlands retraction in a work about Bun_G is not surprising.

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